Continuity of entropies via integral representations

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Introduction. In a series of recent works, a new integral formula for Umegaki's quantum relative entropy was discovered. Originally found by Frenkel [12] and later refined by Jenčova [18] and Hirche and Tomamichel [15], it can be written for states ρ, σ as

$$D(\rho\|\sigma) = (\log e) \int_1^\infty \mathrm{d}\gamma \left(\frac{1}{\gamma} E_\gamma(\rho\|\sigma) + \frac{1}{\gamma^2} E_\gamma(\sigma\|\rho) \right), \tag{1}$$

where Umegaki's quantum *relative entropy* [27] is given by $D(\rho\|\sigma) \coloneqq \operatorname{Tr}\rho\left(\log\rho - \log\sigma\right)$, and the *hockey-stick divergences* are defined by $E_\gamma(\rho\|\sigma) \coloneqq \operatorname{Tr}(\rho - \gamma\sigma)_+$. The quantum relative entropy can be seen as a parent quantity for quantum entropies, as other information measures such as the von Neumann entropy, the conditional entropy, and the mutual information can all be expressed in terms of it. Based on the integral representation (1) and basic mathematical properties of the hockey-stick divergences, we present a conceptually novel and direct quantum technique to derive improved uniform continuity bounds for information measures based on quantum entropy.

Our main general result is a ***dimension-independent semi-continuity relation for the quantum relative entropy with respect to the first argument***. Using it, we derive a wealth of results with numerous applications throughout quantum information theory: (1) a tight continuity relation for the conditional entropy in the case where the two states have equal marginals on the conditioning system, which resolves a conjecture by Wilde [32, Eq. (58)] in this special case; (2) a stronger version of the Fannes–Audenaert inequality on quantum entropy; (3) better estimates on the quantum capacity of approximately degradable channels; (4) an improved continuity relation for the entanglement cost; (5) general upper bounds on asymptotic transformation rates in infinite-dimensional entanglement theory; and (6) a proof of a conjecture due to Christandl, Ferrara, and Lancien on the continuity of 'filtered' relative entropy distances [7, Conjecture 7].

Main result. Our bound can be seen as a fully quantum extension of Csiszár's tight continuity of entropy: For two probability distributions P_X, Q_X on a finite alphabet $\mathcal{X}$ with $\frac{1}{2}\|P_X - Q_X\|_1 \leq \varepsilon \leq 1 - \frac{1}{|\mathcal{X}|}$ in variational distance, one has that

$$\left|H(X)_P - H(X)_Q\right| \leq \varepsilon \log\left(|\mathcal{X}| - 1\right) + h_2(\varepsilon), \tag{2}$$

where $H(X)_P \coloneqq -\sum_{x\in\mathcal{X}} P(x)\log P(x)$ denotes the Shannon entropy, and $h_2(p) \coloneqq p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$ is the *binary entropy function*. Namely, the general form of our main result is as follows.

Theorem 1. *For states ρ, σ, ω on the same system, we have that*

$$D(\rho\|\omega) - D(\sigma\|\omega) \leq \varepsilon \log(M-1) + h_2(\varepsilon), \tag{3}$$

whenever $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1 - \frac{1}{M}$ with M any number such that the operator inequality $\rho \leq M\omega$, meaning that $M\omega - \rho$ is positive semi-definite, holds true.

Note that the left-hand side of (3) does not contain an absolute value. This feature, which makes ours a *semi*-continuity bound rather than a plain continuity bound like (2), is rather fundamental, as the expression on the left-hand side can happen to diverge to $-\infty$. Eq. (3) is in fact tight, in the sense that for all $M \geq 1$ and $\varepsilon \in \left[0, 1 - \frac{1}{M}\right]$ one can find a triple of states ρ, σ, ω obeying the above conditions and saturating (3).

Proof ideas. The conceptual novelty of our approach is that it works ***directly at the level of quantum entropies***, without first operating a reduction to the classical case. This is on the one hand different from all previous approaches to prove quantum (unconditional) extensions of (2) as in [3, 10, 33], and on the other hand it gives improved bounds compared to the only previous quantum approach due to Alicki–Fannes [2] for the conditional case — that has been explored extensively for information measures in quantum information theory (see, e.g., [4]). The proof hinges on the new integral representation (1) and makes extensive use of various properties of the hockey-stick divergences $E_\gamma(\rho\|\sigma)$, such as their variational representation, the monotonicity in γ, their relation to the max-relative entropy, the triangle inequalities, as well as the convexity in γ (see, e.g., [15] for a reference).

Crucially, the integrals from (1) have to be split up into different parts of which each is estimated differently.

Implications. We immediately get an improved Fannes–Audenaert inequality, treating a quantum extension of (2).

Corollary 2. *For states ρ, σ with $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1 - \frac{1}{d\,\lambda_{\max}(\sigma)}$ on a d-dimensional quantum system with $\lambda_{\max}(\sigma)$ denoting the maximal eigenvalue of σ, we have*

$$S(\rho) - S(\sigma) \leq \varepsilon \log\left(d\,\lambda_{\max}(\sigma) - 1\right) + h_2(\varepsilon)\,. \tag{4}$$

The next logical step is to treat *conditional* quantum entropies. For bipartite quantum states, we are able to give a ***tight continuity bound on the conditional entropy*** $H(A|B)_\rho := H(AB)_\rho - H(B)_\rho$ that applies to all pairs of states ρ_{AB}, σ_{AB} with equal marginals on B, i.e., such that $\rho_B = \sigma_B$. By a fortunate coincidence, precisely this special case that turns out to have wide applicability in quantum Shannon theory. (In fact, we could identify only one problem where one needs a general statement, the continuity of the squashed entanglement.) To proceed we need the mixed state Schmidt number [26], $\mathrm{SN}(\rho_{AB}) := \inf_{\rho=\sum_i p_i\psi_i} \max_i \mathrm{SN}\big(|\psi_i\rangle_{AB}\big)$, where the minimization is over pure state convex decompositions $\rho_{AB} = \sum_i p_i\,|\psi_i\rangle\!\langle\psi_i|_{AB}$, and $\mathrm{SN}\big(|\psi_i\rangle_{AB}\big)$ denotes the rank of the reduced state $\mathrm{Tr}_B\,|\psi_i\rangle\!\langle\psi_i|_{AB}$.

Corollary 3. *For states ρ_{AB}, σ_{AB} with $\frac{1}{2}\|\rho - \sigma\|_1 \leq 1 - \frac{1}{|A|\,\mathrm{SN}(\rho_{AB})}$ and $\rho_B = \sigma_B$, we have*

$$H(A|B)_\rho - H(A|B)_\sigma \leq \varepsilon \log\big(|A| \cdot \mathrm{SN}(\rho_{AB}) - 1\big) + h_2(\varepsilon)\,, \tag{5}$$

where $|A|$ denotes the dimension of the A-system.

This pleasantly reproduces the tight classical result [1] as well as the corresponding quantum-classical bound [32] (both of which, however, equally hold even when $\rho_B \neq \sigma_B$). For the fully quantum case we immediately get

$$\big|H(A|B)_\rho - H(A|B)_\sigma\big| \leq \varepsilon \log\big(|A|^2 - 1\big) + h_2(\varepsilon)\,, \tag{6}$$

which proves the quantum conjecture of Wilde [32] for the special case of $\rho_B = \sigma_B$. We note that (5) is then ***exactly tight*** in every (finite) dimension, and as such it improves on the previous state-of-the-art Alicki–Fannes–Winter bound [33],

$$\big|H(A|B)_\rho - H(A|B)_\sigma\big| \leq \varepsilon \log |A|^2 + (1+\varepsilon) \cdot h_2\left(\frac{\varepsilon}{1+\varepsilon}\right)\,. \tag{7}$$

Applications. Plentiful of correlation measures in quantum information theory are based on quantum entropy, and, interestingly, in many cases the marginal constraint $\rho_B = \sigma_B$ is naturally fulfilled. By applying our results and expanding our proof techniques, we get a plethora of tightened continuity estimates for operationally relevant quantities, solving various conjectures in the literature. These include:

1. **Entanglement cost.** For bipartite states ρ_{AB}, a fundamental measure of entanglement is given by the entanglement cost $E_c(\rho)$, i.e., the minimal number of units of entanglement ('ebits') that are required to prepare each copy of ρ_{AB} via local operations assisted by classical communication in the many-copy limit. The quantity E_c can be calculated as [14]

$$E_c(\rho) = \lim_{n\to\infty} \frac{1}{n} E_f\big(\rho^{\otimes n}\big)\,, \quad \text{with} \quad E_f(\rho) := \inf_{\{p_x,\psi_x\}_x:\ \sum_x p_x \psi_x^{AB} = \rho_{AB}} \sum_x p_x\, S\big(\psi_x^A\big)\,, \tag{8}$$

and we then give improved continuity relations for $E_c(\rho)$. Contrary to relations for $E_f(\rho)$, the crucial point is that we need a *quantum* conditional system as in Corollary 3 to control the regularization.

2. **Approximate degradability.** The capacity $Q(\mathcal{N})$ of channels for sending quantum information is in general unknown, but upper bounds can be found using the concept of so-called approximate degradabile channels [19, 25]. As continuity plays a central quantitative role in such estimates, we improve on previous such bounds and, e.g., show that for an ε-degrading channel for $\mathcal{N}$ with $\varepsilon \leq 1 - \frac{1}{|E|^2}$, where $|E|$ is the dimension of the environment of a Stinespring dilation of $\mathcal{N}$, we have

$$I_c(\mathcal{N}) \leq Q(\mathcal{N}) \leq I_c(\mathcal{N}) + \varepsilon \log\left[\big(|E|-1\big)^2\big(|E|+1\big)\right] + 2h_2(\varepsilon)\,. \tag{9}$$

Here, $I_c(\mathcal{N})$ denotes the single-letter coherent information of the channel. We further note that our methods can be combined with our recent improvement, such as, e.g., [17], to arrive at the overall tightest known bounds.

3. **Filtered entropies.** We completely resolve a conjecture of Christandl, Ferrara, and Lancien [7, Conjecture 7]. Namely, we can quantitatively relate under minimal assumptions the continuity of so-called filtered relative entropy distances to the corresponding filtered one-norm distances. Whereas this property is natural to expect, it evaded previous proof techniques, and on a technical level we need to derive an additional lemma to bound the filtered relative entropy distance to the next free states in resource theories (inspired from [21, Proposition 3]).

4. **Transformation rates for infinite-dimensional systems.** We extend known result in entanglement theory to the infinite-dimensional setting and show that the asymptotic transformation rate of bipartite quantum states $\rho_{AB} \to \sigma_{A'B'}$, i.e. the maximum rate of production of copies of σ that can be achieved by consuming copies of ρ_{AB} and using only local operations and classical communication (LOCC) [6], is upper bounded as

$$R_{\mathrm{LOCC}}(\rho \to \sigma) \leq \frac{E_R^{\infty}(\rho)}{E_R^{\infty}(\sigma)}, \tag{10}$$

where the right-hand side features the regularized relative entropy of entanglement [28, 29]. Crucially, this works under the minimal assumption that logarithmic robustness of entanglement [9, 24, 31] of the target stated is finite. The difficulty with (10) is that its standard proof [16, XV.E.2] makes use of asymptotic continuity, a strong form of continuity that does not apply to infinite-dimensional systems. Our main contribution in this section is to generalize (10) to the infinite-dimensional setting, thus avoiding the 'asymptotic continuity catastrophe' described in [11]. This is possible because our fundamental inequality (3) is dimension independent.

Discussion. We presented a flexible and powerful novel proof technique for deriving continuity bounds for correlation measures in quantum information theory. Whereas improvements such as from (6) to the exactly tight (7) are pleasing and have various applications in quantum information theory, we believe that the conceptual strength and novelty of our submission lies in particular in our fully quantum approach, based on Frenkel's integral representation. There are also plentiful of natural follow-up questions to explore. First and foremost, we would like to get rid of the marginal constraint in Corollary 3, which, however, is already highly non-trivial classically [1]. Starting from classical considerations, better understanding quantum conditional majorization appears crucial [5, 13, 30] and it does not seem to be enough to work with divergence centers (as previously done in the Alicki–Fannes approach [22]). More generally, we might ask about a tight continuity relation for the mutual information $I(A:B)_\rho := H(A)_\rho - H(A|B)_\rho$, which to the best of our knowledge is even open classically. We might conjecture that

$$\left|I(A:B)_\rho - I(A:B)_\sigma\right| \overset{?}{\leq} \varepsilon \log\Big(\min\left\{|A|^2, |B|^2\right\} - 1\Big) + h_2(\varepsilon), \tag{11}$$

which would the lead to provably tight bounds on channel capacities, improving on [20, 23]. Extensions to conditional mutual information would be relevant for squashed entanglement [2, 8, 23, 33]. Further, it would lead to (even more) improved approximate degradability bounds on channel capacities in the spirit of [19, 25]. Next, for infinite-dimensional extension of our main Theorem 1 it seems crucial to give a *smoothed* max-relative entropy version to improve on finite-energy bounds from [33, 34]. Lastly, and similarly as for the Alicki–Fannes technique, one might explore extensions, e.g., to Rényi divergences [4].

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