

Entanglement monogamy via multivariate trace inequalities

Mario Berta, Marco Tomamichel

CMP (2024), arXiv:2304.14878

QIP Taipei 01/16/2024

RWTHAACHEN
UNIVERSITY

Imperial College
London



NUS
National University
of Singapore

Matrix analysis for quantum entropy inequalities

A yellow triangular graphic is located in the bottom right corner of the slide, pointing towards the center.

Entropy inequalities

- Probability distribution P_{ABC} with entropy $H(A)_P := -\sum_x P_A^x \log P_A^x \geq 0$
- Mutual information:

$$I(A:B)_P := H(A)_P + H(B)_P - H(AB)_P \geq 0$$

- Conditional entropy:

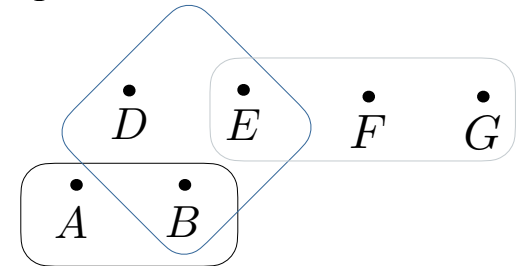
$$H(A|B)_P := H(AB)_P - H(B)_P \geq 0$$

- Conditional mutual information:

$$I(A:B|C)_P := H(AC)_P + H(BC)_P - H(C)_P - H(ABC)_P \geq 0$$

- Non-Shannon type inequalities (!) – entropy cone, e.g.,

$$2I(C:D)_P \leq I(A:B)_P + I(A:CD)_P + 3I(C:D|A)_P + I(C:D|B)_P$$



Quantum entropy inequalities

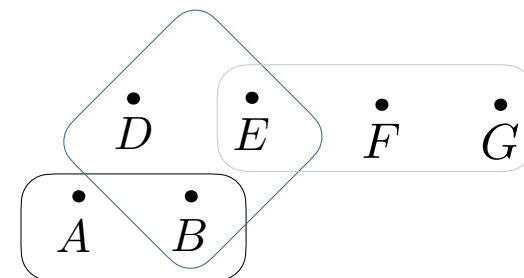
- State $\rho_{ABC} \succcurlyeq 0$, $\text{Tr}[\rho_{ABC}] = 1$ with entropy $H(A)_\rho := -\text{Tr}[\rho_A \log \rho_A] \geq 0$
- $I(A:B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho \geq 0$
- $H(A|B)_\rho := H(AB)_\rho - H(B)_\rho$ can get negative
- Strong subadditivity (SSA):

$$I(A:B|C)_\rho := H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho \geq 0$$

- Equivalent to entanglement monogamy = weak monotonicity:

$$H(A|C)_\rho + H(A|B)_\rho \geq 0$$

- Today: Prove (novel) quantum entropy inequalities via matrix analysis

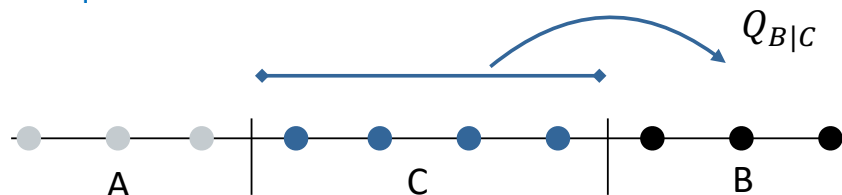


Refined entropy inequalities beyond SSA

SSA refinements

- Recoverability via parent relative entropy $D(P||Q) := \sum_x P^x \log \left(\frac{P^x}{Q^x} \right)$:

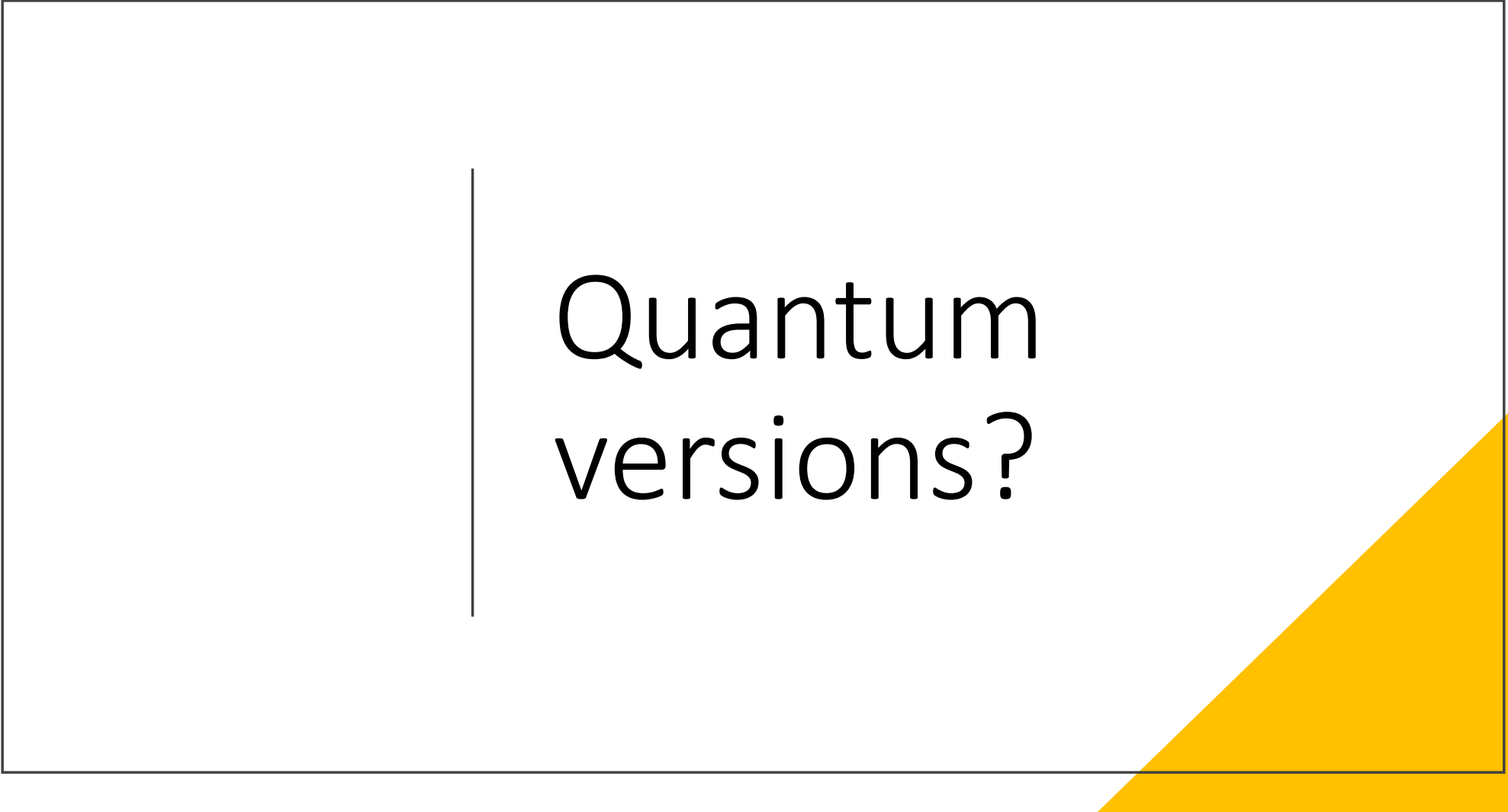
$$I(A:B|C)_P = \inf_{Q_{B|C}} D(P_{ABC} || Q_{B|C} P_{AC}) \geq 0 \text{ with } Q_{BC} = P_{BC} P_C^{-1}$$



- Equivalent Markov chain formulation:

$$I(A:B|C)_P = \inf_{Q_{A-C-B}} D(P_{ABC} || Q_{A-C-B}) \geq 0 \text{ with } Q_{A-C-B} = P_{BC} P_C^{-1} P_{AC}$$

- Equality conditions $I(A:B|C)_P = 0$ from $D(P||Q) = 0 \Leftrightarrow P = Q$



Quantum
versions?

From classical to quantum relative entropy

- Relative entropy: $D(P||Q) = \sum_x P^x \log \left(\frac{P^x}{Q^x} \right)$
- Quantum relative entropy: $D(\rho||\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)]$
- Geometric relative entropy: $\hat{D}(\rho||\sigma) := \text{Tr}[\rho \log(\rho^{1/2} \sigma^{-1} \rho^{1/2})]$
- Measured relative entropy for POVMs $\{M^x\}$ with $\mathbb{P}_{\rho,M}(x) := \text{Tr}[M^x \rho]$ as

$$D_{ALL}(\rho||\sigma) := \sup_M D(\mathbb{P}_{\rho,M}||\mathbb{P}_{\sigma,M})$$

- Good properties, ordering: $\hat{D}(\rho||\sigma) \geq D(\rho||\sigma) \geq D_{ALL}(\rho||\sigma)$

Variant: Locally measured relative entropies

- Bipartite system AB , local measurements with respect to $A:B$ as

$$M = LO(A:B), LOCC_1(A \rightarrow B), LOCC(A:B), SEP(A:B), PPT(A:B)$$

leads to

$$D_M(\rho_{AB} || \sigma_{AB}) := \sup_{M \in M} D(\mathbb{P}_{\rho, M} || \mathbb{P}_{\sigma, M})$$

with $\mathbb{P}_{\rho, M}(x) = \text{Tr}[\rho M^x]$ from POVMs $\{M^x\}$ with $M^x \in M$

- Reasonable properties, ordering:

$$D_{ALL}(\rho_{AB} || \sigma_{AB}) \geq D_{PPT}(\rho_{AB} || \sigma_{AB}) \geq \dots \geq D_{LO}(\rho_{AB} || \sigma_{AB}) \geq 0$$

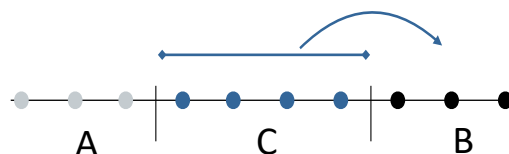
with $D_M(\rho_{AB} || \sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}$ (informationally complete)

A yellow triangle is located in the bottom right corner of the slide, pointing towards the top right.

Quantum recoverability

Quantum SSA refinements: Recoverability

- Non-commutative rewriting:



$$I(A: B|C)_\rho = D(\rho_{ABC} || \exp(\log \rho_{AC} + \log \rho_{BC} - \log \rho_{ABC})) = ???$$

- Recoverability via quantum channels $R_{C \rightarrow BC}$ gives:

a. Equality condition: $I(A: B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = (I_A \otimes R_{C \rightarrow BC})(\rho_{AC})$

b. No-go: $I(A: B|C)_\rho \not\geq \inf_{R_{C \rightarrow BC}} D(\rho_{ABC} || (I_A \otimes R_{C \rightarrow BC})(\rho_{AC}))$

c. Theorem: $I(A: B|C)_\rho \geq \inf_{R_{C \rightarrow BC}} D_{ALL}(\rho_{ABC} || (I_A \otimes R_{C \rightarrow BC})(\rho_{AC}))$

→ QIP 2016 [Fawzi, Renner] + many more, direct matrix analysis proof

QIP 2017 [Sutter, B., Tomamichel]

A yellow right-angled triangle is positioned in the bottom right corner of the slide, with its hypotenuse running from the bottom left towards the top right.

Quantum Markov chains

Quantum SSA refinements: Separability

- Exact quantum Markov chain (QMC) equality condition:

$$I(A:B|C)_\rho = 0 \Leftrightarrow \rho_{ABC} = \bigoplus_k p_k \rho_{AC_L^k}^k \otimes \rho_{C_R^k B}^k \text{ with } C = \bigoplus_k C_L^k \otimes C_R^k$$

- QMC no-go: $I(A:B|C)_\rho \not\geq \inf_{\sigma \in QMC(A-C-B)} D_{ALL}(\rho_{ABC} || \sigma_{ABC})$

- Separability no-go: $I(A:B|C)_\rho \not\geq \inf_{\sigma \in SEP(A:B)} D_{ALL}(\rho_{AB} || \sigma_{AB})$

- Theorem: $I(A:B|C)_\rho \geq \inf_{\sigma \in SEP(A:B)} D_{LOCC_1(B \rightarrow A)}(\rho_{AB} || \sigma_{AB})$

QIP 2011 [Brandão, Christandl, Yard]

→ Today: Give novel matrix analysis proof of strengthened bounds

Proof techniques



Workhorse A: Multivariate trace inequalities

- Multipartite systems $ABC \Leftrightarrow$ non-commuting matrices $\rho_{ABC}, \rho_{AB}, \rho_{BC}, \dots$
- Example trace inequality: Golden-Thompson for H_1, H_2 Hermitian as

$$\text{Tr}[\exp(H_1 + H_2)] \leq \text{Tr}[\exp(H_1) \exp(H_2)]$$

- Multivariate extension for $\{H_k\}_{k=1}^n$ Hermitian as

$$\log \|\exp(\sum_{k=1}^n H_k)\|_p \leq \int_{-\infty}^{\infty} d\beta_0(t) \log \|\prod_{k=1}^n \exp((1 + it)H_k)\|_p$$

with probability distribution $\beta_0(t) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}$

→ QIP 2017 [Sutter, B., Tomamichel] for recoverability SSA strengthening

Workhorse B: Variational formulas

- Relative entropy $D(P||Q) = \sup_{W \geq 0} \sum_x P_x \log W_x - \log(\sum_x Q_x W_x)$

- Quantum relative entropy

$$D(\rho||\sigma) = \sup_{\omega \geq 0} \text{Tr}[\rho \log \omega] - \log \text{Tr}[\exp(\log \sigma + \log \omega)]$$

- Measured relative entropy (Golden-Thompson!)

$$D_{ALL}(\rho||\sigma) = \sup_{\omega \geq 0} \text{Tr}[\rho \log \omega] - \log \text{Tr}[\sigma \omega]$$

- Theorem: Locally measured relative entropy

$$D_M(\rho||\sigma) \leq \sup_{\omega \in C_M} \text{Tr}[\rho \log \omega] - \log \text{Tr}[\sigma \omega]$$

with $C_M := \bigcup_{M \in \mathcal{M}} \text{cone}\{M_x\}$ and equality for $M = LO, LOCC_1(A \rightarrow B)$

Result: SSA separability refinements

- Theorem: $I(A:B|C)_\rho \geq E_{LOCC_1(B \rightarrow A)}(A:B) - \{E(A:C)_\rho - E_{ALL}(A:C)\}$

with shorthand notation relative entropies of entanglement

$$E(A:B) := \inf_{\sigma \in SEP(A:B)} D(\rho_{AB} || \sigma_{AB}), \quad E_M(A:B) := \inf_{\sigma \in SEP(A:B)} D_M(\rho_{AB} || \sigma_{AB})$$

- Corollary: $I(A:B|C)_\rho \geq E_{LOCC_1(B \rightarrow A)}^\infty(A:B) \geq E_{LOCC_1(B \rightarrow A)}(A:B)$

via version on $\rho^{\otimes n}$ and asymptotic achievability

$$D_{ALL}^\infty(\rho || \sigma) := \lim_{n \rightarrow \infty} \frac{1}{n} D_{ALL}(\rho^{\otimes n} || \sigma^{\otimes n}) = D(\rho || \sigma)$$



Proof sketch



Use variational formula

- Theorem: $I(A: B|C)_\rho \geq E_{\text{LOCC}_1(B \rightarrow A)}(A: B) - \{E(A: C)_\rho - E_{\text{ALL}}(A: C)\}$
- Proof via $I(A: B|C)_\rho + E(A: C)_\rho = D(\rho_{ABC} || \exp(\log \rho_{BC} + \log \sigma_{A:C} - \log \rho_C))$

for $\sigma_{A:C} := \operatorname{argmin}_{\sigma \in \text{SEP}(A:C)} D(\rho_{AC} || \sigma_{AC}) \equiv \sum_j \sigma_A^j \otimes \sigma_C^j$

- Variational formula for relative entropy on right-hand side gives

$$\sup_{\omega \geq 0} \operatorname{Tr}[\rho_{ABC} \log \omega_{ABC}] - \log \operatorname{Tr}[\exp(\log \omega_{ABC} + \log \rho_{BC} + \log \sigma_{A:C} - \log \rho_C))]$$

and with choice $\omega_{ABC} = \exp(\log X_{B \rightarrow A} + \log Y_{AC}) \geq 0$ for $X_{B \rightarrow A} \in \text{LOCC}_1(B \rightarrow A)$

$$\sup_{X \in \text{LOCC}_1(B \rightarrow A), Y \geq 0} \operatorname{Tr}[\rho_{AB} \log X_{B \rightarrow A}] + \operatorname{Tr}[\rho_{AC} \log Y_{AC}] - \log \operatorname{Tr}[\exp(5 \text{ summands})]$$

Use multivariate trace inequality

- 5-matrix Golden-Thompson inequality for $p = 2$ leads to:

$$\sup_{X \in LOCC_1(B \rightarrow A), Y \geq 0} \text{Tr}[\rho_{AB} \log X_{B \rightarrow A}] + \text{Tr}[\rho_{AC} \log Y_{AC}] - \log \text{Tr} \left[Y_{AC} \int_{-\infty}^{\infty} d\beta_0(t) X_{B \rightarrow A}^{\frac{1+it}{2}} \gamma_{A:BC}^t X_{B \rightarrow A}^{\frac{1-it}{2}} \right]$$

$$\text{with } \gamma_{A:BC}^t = \sum_j \sigma_A^j \otimes \left(\rho_{BC}^{\frac{1+it}{2}} \rho_C^{\frac{-1-it}{2}} \sigma_C^j \rho_C^{\frac{-1+it}{2}} \rho_{BC}^{\frac{1-it}{2}} \right) \in SEP(A:BC)$$

- With $\gamma_{A:BC} := \int_{-\infty}^{\infty} d\beta_0(t) \gamma_{A:BC}^t$ rewritten as

$$\sup_{X \in LOCC_1(B \rightarrow A), Y \geq 0} \text{Tr}[\rho_{AB} \log X_{B \rightarrow A}] + \text{Tr}[\rho_{AC} \log Y_{AC}] - \log \text{Tr}[X_{B \rightarrow A} \gamma_{A:BC}] \cdot \text{Tr}[Y_{AC} \hat{\gamma}_{A:C}]$$

$$\text{and if } \hat{\gamma}_{A:C} := \frac{\text{Tr}_B \left[\int_{-\infty}^{\infty} d\beta_0(t) X_{B \rightarrow A}^{\frac{1+it}{2}} \gamma_{A:BC}^t X_{B \rightarrow A}^{\frac{1-it}{2}} \right]}{\text{Tr}[X_{B \rightarrow A} \gamma_{A:BC}]} \in SEP(A:C) \text{ this would give the sought-after}$$

$$\text{lower bound } E_{LOCC_1(B \rightarrow A)}(A:B)_\rho + E_{ALL}(A:C)_\rho$$

Use one-way LOCC property

- Altogether: $I(A: B|C)_\rho + E(A: C)_\rho \geq E_{LOCC_1(B \rightarrow A)}(A: B)_\rho + E_{ALL}(A: C)_\rho$

- Comments:

- For the last step $\hat{\gamma}_{A:C} := \frac{\text{Tr}_B \left[\int_{-\infty}^{\infty} d\beta_0 (t) X_{B \rightarrow A}^{\frac{1+it}{2}} \gamma_{A:BC}^t X_{B \rightarrow A}^{\frac{1-it}{2}} \right]}{\text{Tr} [X_{B \rightarrow A} \gamma_{A:B}]} \in SEP(A: C)$ it is crucial that $X_{B \rightarrow A} \in LOCC_1(B \rightarrow A)$ and thus $E_{LOCC_1(B \rightarrow A)}(A: B)_\rho$ appears in lower bound
- Follows as wlog $X_{B \rightarrow A} = \sum_x \omega_A^x \otimes P_B^x$ with $\{P_B^x\}$ set of mutually orthogonal projectors (for B large enough), phases $(1 \pm it)/2$ from multivariate cancel!
- Inequalities used are:
 - i. 5-matrix Golden-Thompson
 - ii. Interaction Hamiltonian $\omega_{ABC} = \exp(\log X_{B \rightarrow A} + \log Y_{AC})$ with $X_{B \rightarrow A} \in LOCC_1(B \rightarrow A)$



Extensions



Further results

- Generalized entanglement monogamy inequalities, e.g.,

$$E(A:BC)_\rho \geq E_{LOCC_1(B \rightarrow A)}(A:B) + E_{ALL}(A:C)_\rho$$

- Conditional entanglement of mutual information (CEMI) for $\rho_{A\bar{A}B\bar{B}}$ gives:
 - Strengthened separability refinement $E_{SEP}(A:B)_\rho$
 - *PPT* separability refinements $P_{PPT}(A:B)_\rho$
- Multipartite separability refinements via n -matrix Golden-Thompson
- Multipartite squashed entanglement and multipartite CEMI lead to corresponding recoverability refinements

Conclusions

- Extending classical entropy inequalities to quantum states gives **entanglement monogamy inequalities**
 - reveals the structure of multipartite quantum states
- Good to have different proofs of results (some proofs might be flawed!)
 - matrix analysis directly leads to tight and all most general bounds
- Enablers:
 - 1) Locally measured quantum relative entropies
 - 2) Variational formulas for quantum entropies
 - 3) Multivariate matrix trace inequalities

Where to go from here

- Entropic uncertainty relations for sets of multiple measurements?
- Faithfulness of multipartite squashed entanglement?
- Quantum entropy versus exact quantum Markov chains (QMC)?
- Quantum entropy cone?
- Upcoming at QIP 2023 (list incomplete):
 - *56 Quantum Rényi and f -divergences from integral representations*
 - *67 Connecting entanglement distillation and entanglement testing with restricted measurements*
 - *85 The Quantum Entropy Cone near its Apex*
 - *96 Clustering of conditional mutual information and quantum Markov structure at arbitrary temperatures*
 - *449 A new operator extension of strong subadditivity of quantum entropy*



I am hiring

- Postdoc and PhD positions at RWTH Aachen University
- Get in contact:
berta@physik.rwth-aachen.de
- **Thank you, questions?**

arXiv:2304.14878