

Open question:  
Is there a way of making entanglement  
theory reversible?

On a gap in the proof of the generalised quantum Stein's lemma  
and its consequences for the reversibility of quantum resources

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# Resource theory of entanglement

- Free states are given by convex hull of product states

$$S_{A:B} := \text{conv}\{|\psi_A\rangle\langle\psi_A| \otimes |\phi\rangle\langle\phi|_B : |\psi\rangle_A \in H_A, |\phi\rangle_B \in H_B\}$$

- Measure global resource robustness  $R_S(\rho_{AB}) := \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}\}$
- Largest set of free operations for transformations  $\rho_{AB} \rightarrow \omega_{AB}$  are  $\delta$ -non-entangling operations (ANE)

$$NE_{\delta(A:B \rightarrow A':B')} := \{\Lambda \in CPTP(AB \rightarrow A'B') : R_S(\Lambda(\sigma_{AB})) \leq \delta \ \forall \sigma_{AB} \in S_{A:B}\}$$

- Distillable entanglement / entanglement cost under ANE (asymptotic rates)
- Open question: Do we have the reversibility  $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$  ?

# Asymptotic characterization of entanglement

- Entanglement cost [Brandão & Plenio, CMP 10], [Datta, IEEE 09]

$$E_C^{ANE}(\rho) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^{\varepsilon}(\rho^{\otimes n} || \sigma^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n) \neq \min_{\sigma \in S} D(\rho || \sigma)$$

- Distillable entanglement [Brandão & Plenio, CMP 10]

$$E_D^{ANE}(\rho) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \beta_{\varepsilon}(\rho^{\otimes n}, S^n)$$

for the hypothesis testing  $\beta_{\varepsilon}(\rho^{\otimes n}, S^n) := \inf_{0 \leq M_n \leq 1} \{ \sup_{\sigma^n \in S^n} \text{Tr}[M_n \sigma^n] : \text{Tr}[(1 - M_n) \rho^{\otimes n}] \leq \varepsilon \}$

- Composite quantum hypothesis testing question  $-\frac{1}{n} \log \beta_{\varepsilon}(\rho^{\otimes n}, S^n) \rightarrow \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n)$ ?

# Reduction to hypothesis testing

- Question if  $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$  reduces to composite quantum hypothesis question

$$-\frac{1}{n} \log \beta_\varepsilon(\rho, S^n) \rightarrow \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n) ?$$

- Converse direction by standard arguments [Brandão & Plenio, CMP 10]

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \beta_\varepsilon(\rho, S^n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n)$$

- [B. *et al.*, arXiv 22] recently found that achievability direction “ $\geq$ ” remains open
- Needed: Composite hypothesis test that asymptotically achieves regularized relative entropy of entanglement!

# Known proof techniques

- I. via Petz-Rényi divergences
- II. via measured divergence
- III. via smooth max-relative entropy
- IV. find counterexamples to conjecture

# Some starting references (incomplete)

- B., Brandão, Hirche: CMP 385, 55 (2021)
- B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel: arXiv:2205.02813 (2022)
- Audenaert, Nussbaum, Szkola, Verstraete: CMP 279, 251 (2008)
- Brandão, Harrow, Lee, Peres: IEEE Trans. 66, 5037 (2020)
- Mosonyi, Szilágyi, Weiner: arXiv:2011.04645 (2021)
- Lami & Regula: arXiv:2111.02438 (2021)
- Bergh, Datta, Salzmann, Wilde: arXiv:2206.08350 (2022)

# I. Universal hypothesis tests via Petz-Rényi divergences

- Sion minimax + Audenaert inequality for  $s \in (0,1)$  gives [Audenaert *et al.*, CMP 08]

$$-\frac{1}{n} \log \beta_\varepsilon(\rho^{\otimes n}, S^n) = -\frac{1}{n} \sup_{\sigma^n \in S^n} \inf_{0 \leq M_n \leq 1, \text{Tr}[M_n \rho^{\otimes n}] \geq 1-\varepsilon} \log \text{Tr}[M_n \sigma^n] \geq \frac{1}{n} \inf_{\sigma^n \in S^n} D_s(\rho^{\otimes n} || \sigma^n) - \frac{1}{n} \cdot \frac{s}{1-s} \log \frac{1}{\varepsilon}$$

for the additive  $D_s(\rho || \sigma) := \frac{1}{s-1} \log \text{Tr}[\rho^s \sigma^{1-s}]$  with  $\lim_{s \rightarrow 1} D_s(\rho || \sigma) = D(\rho || \sigma)$

- Single-letter: de Finetti, take limits (i)  $n \rightarrow \infty$  (ii)  $\varepsilon \rightarrow 0$  (iii)  $s \rightarrow 1$  in order [B. et al., CMP 21]
- Generally, with information variance  $V(\rho || \sigma) := \text{Tr}[\rho(\log \rho - \log \sigma - D(\rho || \sigma))^2]$  to bound

$$\frac{1}{n} |D_s(\rho^{\otimes n} || \sigma^n) - D(\rho^{\otimes n} || \sigma^n)| \leq \frac{s-1}{2} \cdot \frac{V(\rho^{\otimes n} || \sigma^n)}{n} + \frac{O((s-1)^2)}{n}$$

- [Brandão & Plenio, CMP 10] claimed that  $V(\rho^{\otimes n} || \sigma^n) \leq o(2^{-n})$ , but already

$$V(\rho^{\otimes n} || \sigma^{\otimes n}) = n \cdot V(\rho || \sigma) \not\leq o(2^{-n}) \quad \rightarrow \text{Remains open: de Finetti / Schur-Weyl duality?}$$

## II. Universal hypothesis tests via measured divergence

- Measured relative entropy [Donald, CMP 86] with [Brandão *et al.*, IEEE 20]

$$D_M(\rho||\sigma) := \sup_M D_{KL}(M(\rho)||M(\sigma)) \text{ with } \inf_{\rho \in T, \sigma \in S} D_M(\rho||\sigma) = \sup_M \inf_{\rho \in T, \sigma \in S} D_{KL}(M(\rho)||M(\sigma))$$

- (i) measure, (ii) apply classical composite hypothesis result, (iii) use asymptotic achievability of measured relative entropy for  $\rho^n, \sigma^n$  permutation invariant [B. *et al.*, CMP 21]

$$\frac{1}{n} D_M(\rho^n||\sigma^n) \rightarrow \frac{1}{n} D(\rho^n||\sigma^n) \text{ for } n \rightarrow \infty$$

- Gives pseudo-entanglement theory and pseudo-entanglement in blocks [B. *et al.*, arXiv 22]
- Entanglement theory remains open!
- Alternatively, one has [Brandão *et al.*, IEEE 20]

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D_{M_{SEP}}(\rho^{\otimes n}||\sigma^n) \rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n}||\sigma^n) ?$$

# III. Universal hypothesis tests via max-relative entropy

- For  $\varepsilon \in (0,1)$  we have [Anshu *et al.*, JMP 19]

$$-\frac{1}{n} \log \sup_{\sigma^n \in S^n} \beta_\varepsilon(\rho^{\otimes n}, \sigma^n) \geq \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^{\sqrt{1-\varepsilon}}(\rho^{\otimes n} || \sigma^n) - \frac{1}{n} \log \frac{1}{\varepsilon}$$

- Previously mentioned asymptotic equipartition property (AEP) for max-relative entropy

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^\delta(\rho^{\otimes n} || \sigma^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n) \text{ not enough as no strong converse!}$$

- Similar open problems:
  - Quantum channel AEP [Gour & Winter, PRL 19]
  - Strong converse channel discrimination & channel capacities [Fang *et al.*, arXiv 21] [Bergh *et al.*, arXiv 21]
  - Stronger entropy accumulation [Metger *et al.*, arXiv 22]

# IV. Lami & Regula arXiv:2111.02438

- Title: No second law of entanglement manipulation after all
- Recall:
  - $\delta$ -non-entangling operations  $NE_{\delta(AB \rightarrow A'B')} = \{\Lambda \in CPTP(AB \rightarrow A'B') : R_S(\Lambda(\sigma_{AB})) \leq \delta \forall \sigma_{AB} \in S_{A:B}\}$
  - with global resource robustness  $R_S(\rho_{AB}) = \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}\}$
- Replace  $R_S(\rho_{AB})$  with resource robustness

$$\bar{R}_S(\rho_{AB}) := \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}, \sigma_{AB} \in S_{A:B}\} \geq R_S(\rho_{AB})$$

and correspondingly

$$\overline{NE}_{\delta(AB \rightarrow A'B')} := \{\Lambda \in CPTP(AB \rightarrow A'B') : \bar{R}_S(\Lambda(\sigma_{AB})) \leq \delta \forall \sigma_{AB} \in S_{AB}\}$$

- Main result: there exists quantum state  $\rho$  with  $E_D^{\overline{ANE}}(\rho) < E_C^{\overline{ANE}}(\rho)$

# Is there a way of making entanglement theory reversible?

→ Becomes composite quantum hypothesis testing question

$$-\frac{1}{n} \log \beta_\varepsilon(\rho^{\otimes n}, S^n) \geq \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n)?$$

with  $\beta_\varepsilon(\rho^{\otimes n}, S^n) := \inf_{0 \leq M_n \leq 1} \{ \sup_{\sigma^n \in S^n} \text{Tr}[M_n \sigma^n] : \text{Tr}[(1 - M_n) \rho^{\otimes n}] \leq \varepsilon \}$

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