



On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources

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Based on

- **On composite quantum hypothesis testing**

B., Brandão, Hirche
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- **On a gap in the proof of the generalised quantum Stein's lemma and its consequences for the reversibility of quantum resources**

B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel
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Outline

- Quantum hypothesis testing
- Quantum resource theories
- Asymptotic reversibility?
- Proof techniques
- Conclusion

Quantum hypothesis testing

Symmetric quantum hypothesis testing

- Two sequences ρ_n, σ_n on $H^{\otimes n}$, discriminate them with two outcome POVM $\{M_n, (1 - M_n)\}$
- Two types of errors:

$$\alpha^n(M_n) := \text{Tr}[\rho_n(1 - M_n)] \text{ Type 1} \quad \text{and} \quad \beta^n(M_n) := \text{Tr}[\sigma_n M_n] \text{ Type 2}$$

- Asymptotic independent and identically distributed (IID) for $\rho_n = \rho^{\otimes n}, \sigma_n = \sigma^{\otimes n}$
- Symmetric setting

$$\xi(\rho^{\otimes n}, \sigma^{\otimes n}) := \inf_{0 \leq M_n \leq 1} \frac{\alpha^n(M_n)}{2} + \frac{\beta^n(M_n)}{2}$$

gives **quantum Chernoff bound** [Audenaert *et al.*, PRL 07]

$$\xi(\rho, \sigma) := \lim_{n \rightarrow \infty} -\frac{\log \xi(\rho^{\otimes n}, \sigma^{\otimes n})}{n} = -\log \min_{0 \leq s \leq 1} \text{Tr}[\rho^s \sigma^{1-s}]$$

Asymmetric quantum hypothesis testing

- Asymptotic IID $\rho_n = \rho^{\otimes n}, \sigma_n = \sigma^{\otimes n}$ asymmetric setting

$$\beta_\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n}) := \inf_{0 \leq M_n \leq 1} \{\beta^n(M_n) : \alpha^n(M_n) \leq \varepsilon\}$$

gives **quantum Stein's lemma** [Hiai & Petz, CMP 91]

$$\beta(\rho, \sigma) := \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} - \frac{\log \beta_\varepsilon(\rho^{\otimes n}, \sigma^{\otimes n})}{n} = D(\rho || \sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)]$$

- Fundamental tasks in quantum statistics, underlying much of quantum information theory
- What about composite hypotheses? That is,

$$\rho^{\otimes n} \text{ with } \rho \in T \quad \text{versus} \quad \sigma^{\otimes n} \text{ with } \sigma \in S?$$

Composite hypothesis testing

- Asymptotic IID $\rho^{\otimes n}$ with $\rho \in T$ versus $\sigma^{\otimes n}$ with $\sigma \in S$, asymmetric setting

$$\beta_\varepsilon(T^n, S^n) := \inf_{0 \leq M_n \leq 1} \{ \sup_{\sigma \in S} \text{Tr}[M_n \sigma^{\otimes n}] : \sup_{\rho \in T} \text{Tr}[(1 - M_n) \rho^{\otimes n}] \leq \varepsilon \}$$

- Thought-after characterization

$$\beta(T, S) := \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} - \frac{\log \beta_\varepsilon(T^n, S^n)}{n} = ?$$

- For $\rho \in T, \sigma \in S$ pairwise commuting, **composite Stein's lemma** [Leviton & Merhav, IEEE 02]

$$\beta(T, S) = \inf_{P \in T} \inf_{Q \in S} \beta(P, Q) = \inf_{P \in T} \inf_{Q \in S} D_{KL}(P || Q) \text{ with } D_{KL}(P || Q) := \sum_x p_x \log \frac{p_x}{q_x}$$

for eigendistributions P, Q in common eigenbasis of ρ, σ

- What about fully quantum version?

Composite quantum hypothesis testing

- Partial results for special cases:

[Hayashi, JPA 02], [Bjelaković *et al.*, CMP 05], [Brandão & Plenio, CMP 10], [Hayashi & Tomamichel, JMP 16], etc.

- **Composite quantum Stein's lemma** for T, S convex [B. et al., CMP 21]

$$\beta(T, S) = \lim_{n \rightarrow \infty} \frac{1}{n} \inf_{\rho \in T} \inf_{\mu \in \text{Meas}(S)} D\left(\rho^{\otimes n} \parallel \int \sigma^{\otimes n} d\mu(\sigma)\right) \neq \inf_{\rho \in T} \inf_{\sigma \in S} D(\rho \parallel \sigma) \text{ in general,}$$

see also [Mosonyi *et al.*, arXiv 21], as one does not have the quantum entropy inequality

$$D\left(\rho^{\otimes n} \parallel \int \sigma^{\otimes n} d\mu(\sigma)\right) \not\geq n \cdot \inf_{\sigma \in S} D(\rho \parallel \sigma)$$

- Nevertheless, various examples of interest do become single-letter anyway

Quantum resource theories

Resource theory of entanglement

- All also works for general resource theories (under suitable axiom set)
- Free states are separable states on $H_{AB} := H_A \otimes H_B$, that is, convex hull of product states

$$S_{A:B} := \text{conv}\{|\psi_A\rangle\langle\psi_A| \otimes |\phi\rangle\langle\phi|_B : |\psi\rangle_A \in H_A, |\phi\rangle_B \in H_B\}$$

+ all other states are entangled (i.e., resourceful)

- Unit is ebit $\Phi_{AB} := |\Phi\rangle\langle\Phi|_{AB}$ with $|\Phi\rangle_{AB} := \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$
- Entanglement measure: $R_S(\rho_{AB}) := \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}\}$ **global resource robustness**
- Free operations for transformations $\rho_{AB} \rightarrow \omega_{AB}$? The largest meaningful such set is given by **δ -non-entangling operations**

$$NE_{\delta(A:B \rightarrow A':B')} := \{\Lambda \in \text{CPTP}(AB \rightarrow A'B') : R_S(\Lambda(\sigma_{AB})) \leq \delta \ \forall \sigma_{AB} \in S_{A:B}\}$$

Asymptotic resource theory of entanglement

- Distillable entanglement under asymptotically non-entangling operations (ANE):

$$E_D^{ANE}(\rho) := \sup_{(k_n), (\delta_n)} \left\{ \liminf_{n \rightarrow \infty} \frac{k_n}{n} : \lim_{n \rightarrow \infty} \min_{\Lambda \in NE_{\delta_n}} \|\Lambda(\rho^{\otimes n}) - \Phi^{\otimes k_n}\|_1 = 0, \lim_{n \rightarrow \infty} \delta_n = 0 \right\}$$

- Entanglement cost under ANE:

$$E_C^{ANE}(\rho) := \inf_{(k_n), (\delta_n)} \left\{ \limsup_{n \rightarrow \infty} \frac{k_n}{n} : \lim_{n \rightarrow \infty} \min_{\Lambda \in NE_{\delta_n}} \|\Lambda(\Phi^{\otimes k_n}) - \rho^{\otimes n}\|_1 = 0, \lim_{n \rightarrow \infty} \delta_n = 0 \right\}$$

- Asymptotic **transformation rate** $\rho_{AB} \rightarrow \omega_{AB}$ under ANE:

$$R^{ANE}(\rho \rightarrow \omega) := \sup_{(k_n), (\delta_n)} \left\{ \liminf_{n \rightarrow \infty} \frac{k_n}{n} : \lim_{n \rightarrow \infty} \min_{\Lambda \in NE_{\delta_n}} \|\Lambda(\rho^{\otimes n}) - \omega^{\otimes k_n}\|_1 = 0, \lim_{n \rightarrow \infty} \delta_n = 0 \right\}$$

- **Asymptotic reversibility?**

Asymptotic characterization of entanglement

- Asymptotically reversible under ANE?

$$R^{ANE}(\rho \rightarrow \omega) \cdot R^{ANE}(\omega \rightarrow \rho) = 1 \quad \text{or in other words} \quad E_D^{ANE}(\rho) = E_C^{ANE}(\rho)?$$

- Entanglement cost [Brandão & Plenio, CMP 10], [Datta, IEEE 09]

$$E_C^{ANE}(\rho) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^{\varepsilon}(\rho^{\otimes n} || \sigma^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n) \neq \min_{\sigma \in S} D(\rho || \sigma)$$

- Distillable entanglement [Brandão & Plenio, CMP 10]

$$E_D^{ANE}(\rho) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \beta_{\varepsilon}(\rho^{\otimes n}, S^n)$$

for the hypothesis testing $\beta_{\varepsilon}(\rho^{\otimes n}, S^n) := \inf_{0 \leq M_n \leq 1} \{ \sup_{\sigma^n \in S^n} \text{Tr}[M_n \sigma^n] : \text{Tr}[(1 - M_n) \rho^{\otimes n}] \leq \varepsilon \}$

- Composite quantum hypothesis testing question $-\frac{1}{n} \log \beta_{\varepsilon}(\rho^{\otimes n}, S^n) \rightarrow \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n)?$

Reduction to hypothesis testing

- Question if $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$ reduces to composite quantum hypothesis question

$$-\frac{1}{n} \log \beta_\varepsilon(\rho, S^n) \rightarrow \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n)?$$

- Converse direction by standard arguments [Brandão & Plenio, CMP 10]:

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \beta_\varepsilon(\rho, S^n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n)$$

- [B. *et al.*, arXiv 22] recently found that **achievability direction “ $\geq$ ” remains open**
- Setting: $T^n = \{\rho^{\otimes n}\}$ singleton, but separable set $S^n \equiv S_{A^n:B^n} \equiv S_{(A_1 \cdots A_n:B_1 \cdots B_n)}$ is **not IID and could be entangled** across different A_i 's and B_i 's, resp.
- Results from [B. *et al.*, CMP 21] do not directly apply!

What can be shown?

- Pseudo-entanglement theory:

$\bar{S}_{A^n:B^n} := \text{conv}\{\bigotimes_{j=1}^n \sigma_{A_j B_j}^{(j)} : \sigma_{A_j B_j}^{(j)} \in S_{A_j B_j} \forall j\}$ separable across the partition $A_1: \dots : A_n: B_1: \dots : B_n$
and combination of [Brandão *et al.*, IEEE 20], [B. et al., CMP 21] gives

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \beta_{\varepsilon}(\rho^{\otimes n}, \bar{S}^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in \bar{S}^n} D(\rho^{\otimes n} || \sigma^n)$$

- Pseudo-entanglement in blocks $A^k := A_1 \dots A_k, B^k := B_1 \dots B_k$ with $\bar{S}_{A^k:B^k}^k$ [B. *et al.*, arXiv 22]

$$\lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{nk} \min_{\sigma^{nk} \in \bar{S}^{n,k}} \log \beta_{\varepsilon}(\rho^{\otimes nk}, \sigma^{nk}) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in \bar{S}^n} D(\rho^{\otimes n} || \sigma^n)$$

- Remains open if $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \min_{\sigma^n \in \bar{S}^n} \log \beta_{\varepsilon}(\rho^{\otimes n}, \sigma^n) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in \bar{S}^n} D(\rho^{\otimes n} || \sigma^n)$?

- **Proof techniques:
Universal hypothesis tests**

- 1) via Petz-Rényi divergences?
- 2) via measured divergence?
- 3) via max-relative entropy?

1) Universal hypothesis tests via Petz-Rényi divergences

- Sion minimax + Audenaert inequality for $s \in (0,1)$ gives [Audenaert *et al.*, CMP 08]

$$-\frac{1}{n} \log \beta_\varepsilon(\rho^{\otimes n}, S^n) = -\frac{1}{n} \sup_{\sigma^n \in S^n} \inf_{0 \leq M_n \leq 1, \text{Tr}[M_n \rho^{\otimes n}] \geq 1-\varepsilon} \log \text{Tr}[M_n \sigma^n] \geq \frac{1}{n} \inf_{\sigma^n \in S^n} D_s(\rho^{\otimes n} || \sigma^n) - \frac{1}{n} \cdot \frac{s}{1-s} \log \frac{1}{\varepsilon}$$

for the additive $D_s(\rho || \sigma) := \frac{1}{s-1} \log \text{Tr}[\rho^s \sigma^{1-s}]$ with $\lim_{s \rightarrow 1} D_s(\rho || \sigma) = D(\rho || \sigma)$

- Single-letter: de Finetti, take limits (i) $n \rightarrow \infty$ (ii) $\varepsilon \rightarrow 0$ (iii) $s \rightarrow 1$ in order [B. et al., CMP 21]
- Generally, with information variance $V(\rho || \sigma) := \text{Tr}[\rho(\log \rho - \log \sigma - D(\rho || \sigma))^2]$ to bound

$$\frac{1}{n} |D_s(\rho^{\otimes n} || \sigma^n) - D(\rho^{\otimes n} || \sigma^n)| \leq \frac{s-1}{2} \cdot \frac{V(\rho^{\otimes n} || \sigma^n)}{n} + \frac{O((s-1)^2)}{n}$$

- [Brandão & Plenio, CMP 10] claimed that $V(\rho^{\otimes n} || \sigma^n) \leq o(2^{-n})$, but already

$V(\rho^{\otimes n} || \sigma^{\otimes n}) = n \cdot V(\rho || \sigma) \not\leq o(2^{-n}) \rightarrow \text{Remains open: de Finetti / Schur-Weyl duality?}$

2) Universal hypothesis tests via measured divergence

- Measured relative entropy [Donald, CMP 86] with [Brandão *et al.*, IEEE 20]

$$D_M(\rho||\sigma) := \sup_M D_{KL}(M(\rho)||M(\sigma)) \text{ with } \inf_{\rho \in T, \sigma \in S} D_M(\rho||\sigma) = \sup_M \inf_{\rho \in T, \sigma \in S} D_{KL}(M(\rho)||M(\sigma))$$

- (i) measure, (ii) apply classical composite hypothesis result, (iii) use asymptotic achievability of measured relative entropy for ρ^n, σ^n permutation invariant [B. *et al.*, CMP 21]

$$\frac{1}{n} D_M(\rho^n||\sigma^n) \rightarrow \frac{1}{n} D(\rho^n||\sigma^n) \text{ for } n \rightarrow \infty$$

- Gives pseudo-entanglement theory and pseudo-entanglement in blocks [B. *et al.*, arXiv 22]
- **Remains open:** entanglement theory. Alternatively, one has [Brandão *et al.*, IEEE 20]

$$\lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D_{M_{SEP}}(\rho^{\otimes n}||\sigma^n) \rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n}||\sigma^n) ?$$

3) Universal hypothesis tests via max-relative entropy

- For $\varepsilon \in (0,1)$ we have [Anshu *et al.*, JMP 19]

$$-\frac{1}{n} \log \sup_{\sigma^n \in S^n} \beta_\varepsilon(\rho^{\otimes n}, \sigma^n) \geq \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^{\sqrt{1-\varepsilon}}(\rho^{\otimes n} || \sigma^n) - \frac{1}{n} \log \frac{1}{\varepsilon}$$

- Previously mentioned asymptotic equipartition property (AEP) for max-relative entropy

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D_{\max}^\delta(\rho^{\otimes n} || \sigma^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n) \text{ not enough as no strong converse!}$$

- Similar open problems:
 - Quantum channel AEP [Gour & Winter, PRL 19]
 - Strong converse channel discrimination & channel capacities [Fang *et al.*, arXiv 21] [Bergh *et al.*, arXiv 21]
 - Stronger entropy accumulation [Metger *et al.*, arXiv 22]

- **Conclusion**

Outlook

- Question if $E_D^{ANE}(\rho) = E_C^{ANE}(\rho)$ reduces to composite quantum hypothesis question

$$-\frac{1}{n} \log \beta_\varepsilon^n(\rho, S^n) \rightarrow \frac{1}{n} \min_{\sigma^n \in S^n} D(\rho^{\otimes n} || \sigma^n) ?$$

This remains open.

- Take step back, classical version of non-IID problem? Not clear, cf. [Mosonyi *et al.*, arXiv 21]
- Composite hypothesis testing will hold for some resource theories under suitable axiom set, but must be shown “manually” every time – and so far, we only have single-letter solutions
- Reversibility of resource theories? **If I had to guess, reversibility does not hold in general**
- Hint for resource theory of entanglement: [Lami & Regula, arXiv 21]

Lami & Regula arXiv:2111.02438

- Title: **No second law of entanglement manipulation after all**
- Recall:
 - δ -non-entangling operations $NE_{\delta(AB \rightarrow A'B')} = \{\Lambda \in CPTP(AB \rightarrow A'B') : R_S(\Lambda(\sigma_{AB})) \leq \delta \forall \sigma_{AB} \in S_{A:B}\}$
 - with global resource robustness $R_S(\rho_{AB}) = \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}\}$
- Replace $R_S(\rho_{AB})$ with resource robustness

$$\bar{R}_S(\rho_{AB}) := \{s \geq 0 : \frac{\rho_{AB} + s\sigma_{AB}}{1+s} \in S_{A:B}, \sigma_{AB} \in S_{A:B}\} \geq R_S(\rho_{AB})$$

and correspondingly

$$\overline{NE}_{\delta(AB \rightarrow A'B')} := \{\Lambda \in CPTP(AB \rightarrow A'B') : \bar{R}_S(\Lambda(\sigma_{AB})) \leq \delta \forall \sigma_{AB} \in S_{AB}\}$$

- Main result: there exists quantum state ρ with $E_D^{\overline{ANE}}(\rho) < E_C^{\overline{ANE}}(\rho)$



Thank you!

- B., Brandão, Hirche: CMP 385, 55 (2021)
- B., Brandão, Gour, Lami, Plenio, Regula, Tomamichel: arXiv:2205.02813 (2022)
- Audenaert, Nussbaum, Szkola, Verstraete: CMP 279, 251 (2008)
- Brandão, Harrow, Lee, Peres: IEEE 66, 5037 (2020)
- Mosonyi, Szilágyi, Weiner: arXiv:2011.04645 (2021)
- Lami & Regula: arXiv:2111.02438 (2021)
- Bergh, Datta, Salzmann, Wilde: arXiv:2206.08350 (2022)