

Characterizing quantum correlations of fixed dimension

Mario Berta

arXiv:2005.08883
with Jee, Sparaciari, Fawzi

IPAM 02/08/2021

Imperial College London
Amazon Web Services

Motivation

- ▶ Entanglement leads to quantum correlations stronger than classical correlations [Bell 64]:

Bell inequalities

Questions

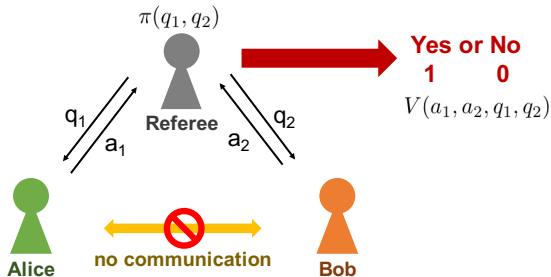
- ▶ How to quantify classical vs quantum correlations?
 - ▶ How does this depend on the underlying dimension?
 - ▶ What is the computational complexity of correlations?
-
- ▶ Quantifying correlations in multipartite systems from computer science perspective:

Non-local games

Non-local games

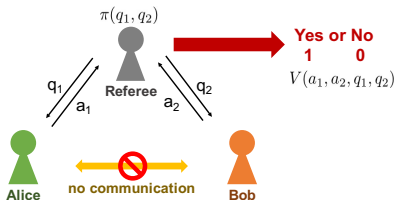
Setting

- Two-prover one-round games:



- Referee gives Alice and Bob **questions** $q_1 \in Q_1$ and $q_2 \in Q_2$ according to the probability distribution $\pi(q_1, q_2)$
- Alice and Bob give **answers** $a_1 \in A_1$ and $a_2 \in A_2$ back to the referee
- Alice and Bob win or lose according to the **rule function**
 $V: A_1 \times A_2 \times Q_1 \times Q_2 \rightarrow \{0, 1\}$

Maximum winning probabilities



- **CHSH game** $|A_1| = |A_2| = |Q_1| = |Q_2| = 2$, $\pi(q_1, q_2) = \frac{1}{4}$ [Clauser *et al.* 69]:

$V_{\text{CHSH}}(a_1, a_2, q_1, q_2) = 1$ if $q_1 \cdot q_2 = a_1 \oplus a_2$, and $V_{\text{CHSH}}(a_1, a_2, q_1, q_2) = 0$ otherwise

- Correlations quantified by respective **maximum winning probabilities**:

$$\omega_C(V, \pi) := \sup_{(e, d)} \sum_{q_1, q_2} \pi(q_1, q_2) \sum_{a_1, a_2} V(a_1, a_2, q_1, q_2) e(a_1|q_1) d(a_2|q_2)$$

$$\omega_{Q(T)}(V, \pi) := \sup_{(E, D, \rho)} \sum_{q_1, q_2} \pi(q_1, q_2) \sum_{a_1, a_2} V(a_1, a_2, q_1, q_2) \text{Tr}[\rho_{T\hat{T}}(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2))]$$

$$\omega_Q(V, \pi) := \sup_{(\mathcal{H}, E, D, \rho)} \sum_{q_1, q_2} \pi(q_1, q_2) \sum_{a_1, a_2} V(a_1, a_2, q_1, q_2) \text{Tr}[\rho_{T\hat{T}}(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2))]$$

Previous results

- ▶ **CHSH game** $\omega_C(V_{\text{CHSH}}) \equiv \omega_C(V_{\text{CHSH}}, 1/4)$ [Clauser *et al.* 69, Tsirelson 80]:

$$\omega_C(V_{\text{CHSH}}) = 0.75 < 0.85 \approx \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) = \omega_{Q(2)}(V_{\text{CHSH}}) = \omega_Q(V_{\text{CHSH}})$$

Hardness:

- ▶ Approx. $\omega_C(V, \pi)$ to constant multiplicative factor NP-hard [Arora *et al.* 98]
- ▶ Approx. $\omega_Q(V, \pi)$ not possible for an algorithm running in finite time [Ji *et al.* 20]

Algorithms:

- ▶ Quasi-polynomial time algorithm to approximate $\omega_C(V, \pi)$ to constant additive error for free games [Aaronson *et al.* 14, Brandão & Harrow 17]
- ▶ Polynomial time algorithms to compute $\omega_Q(V, \pi)$ for XOR [Cleve *et al.* 04] and unique games [Kempe *et al.* 10]
- ▶ NPA hierarchy: strong heuristics to approximate $\omega_Q(V, \pi)$ [Navascués *et al.* 08]

Question

What about $\omega_{Q(T)}(V, \pi)$ for fixed dimension T ?

Quantum correlations of fixed dimension

Quantum correlations of fixed dimension

Motivation:

- ▶ Quantum information processing, e.g., quantum error correction [B. *et al.* 18]
- ▶ Device-independence: **dimension witness** for $\omega_{Q(T)}(V, \pi) < \omega_Q(V, \pi)$

Algorithms:

- ▶ Various heuristics to approximate $\omega_{Q(T)}(V, \pi)$ [Navascués *et al.* 14/15], but known worst case guarantee is **exponential**

Main result

For free games, i.e., $\pi(q_1, q_2) = \pi(q_1) \times \pi(q_2)$, with $|Q_1| = |Q_2| =: Q$ and $|A_1| = |A_2| =: A$, we give an approximation algorithm with complexity

$$\exp\left(\mathcal{O}\left(\frac{T^{12}}{\varepsilon^2} \log(\mathbf{A}T) (\log(\mathbf{Q}) + \log(\mathbf{A}T))\right)\right)$$

to compute additive ε -approximations of $\omega_{Q(T)}(V, \pi)$. That is, for fixed dimension T , the complexity scales **polynomially** in Q and **quasi-polynomially** in A .

Connection to quantum separability problem

$$\omega_{Q(T)}(V, \pi) := \sup_{(E, D, \rho)} \sum_{q_1, q_2} \pi(q_1, q_2) \sum_{a_1, a_2} V(a_1, a_2, q_1, q_2) \text{Tr}[\rho_{T\hat{T}}(E_T(a_1|q_1) \otimes D_{\hat{T}}(a_2|q_2))]$$

$$\omega_{Q(T)}(V, \pi) = |T|^2 \cdot \sup_{(E, D, \rho)} \text{Tr}[(V_{A_1 A_2 Q_1 Q_2} \otimes \Phi_{T\hat{T}|S\hat{S}})(E_{A_1 Q_1 T} \otimes D_{A_2 Q_2 \hat{T}} \otimes \rho_{S\hat{S}})]$$

$$\text{s.t. } \rho_{S\hat{S}} \geq 0, \text{Tr}[\rho_{S\hat{S}}] = 1$$

$$E_{A_1 Q_1 T} = \sum_{q_1, q_1} \pi_1(q_1) |a_1 q_1\rangle \langle a_1 q_1|_{A_1 Q_1} \otimes \frac{E_T(a_1|q_1)}{|T|} \geq 0, E_{Q_1 T} = \sum_{q_1} \pi_1(q_1) |q_1\rangle \langle q_1|_{Q_1} \otimes \frac{\text{id}_T}{|T|}$$

$$D_{A_2 Q_2 \hat{T}} = \sum_{q_2, q_2} \pi_2(q_2) |a_2 q_2\rangle \langle a_2 q_2|_{A_2 Q_2} \otimes \frac{D_{\hat{T}}(a_2|q_2)}{|T|} \geq 0, D_{Q_2 \hat{T}} = \sum_{q_2} \pi_2(q_2) |q_2\rangle \langle q_2|_{Q_2} \otimes \frac{\text{id}_{\hat{T}}}{|T|}$$

- By linearity, equivalently optimize over

$$\sum_i p_i \cdot E_{A_1 Q_1 T}^i \otimes D_{A_2 Q_2 \hat{T}}^i \otimes \rho_{S\hat{S}}^i$$

⇒ tripartite **quantum separability problem** with linear constraints!

- Hard problem [Gharibian 10], but DPS hierarchy [Doherty *et al.* 02] gives approximation algorithms via **quantum de Finetti theorems**

Quantum de Finetti theorems

Monogamous entanglement

- ▶ Quantum states ρ_{AB} are called **n -shareable** on B with respect to A if

$$\rho_{AB_1^n} \equiv \rho_{AB_1 \dots B_n} \text{ with } \rho_{AB_j} = \rho_{AB} \quad \forall j \in [n]$$

$\Rightarrow$ characterizes separable states [Stoermer 69]

Quantum de Finetti

For n -shareable quantum states ρ_{AB} , there exist probabilities $\{p_i\}_{i \in I}$ and quantum states $\sigma_A^i, \omega_B^i \quad \forall i \in I$ such that [Christandl *et al.* 07]

$$\left\| \rho_{AB} - \sum_{i \in I} p_i \cdot \sigma_A^i \otimes \omega_B^i \right\|_1 \leq \frac{|B|^2}{n}.$$

- ▶ **n -shareable is efficient criteria** to check (positive semi-definite) — though $n \gg |B|^2$ needed for good approximation on the set of separable states
- ▶ Tripartite quantum de Finetti with linear constraints?

Adapted quantum de Finetti

Tripartite with linear constraints

For ρ_{ABC} **n -shareable** on B wrt AC^n and n -shareable on C wrt AB^n , there exist probabilities $\{p_i\}_{i \in I}$ and quantum states σ_A^i , ω_B^i and $\gamma_C^i \forall i \in I$ such that

$$\left\| \rho_{ABC} - \sum_{i \in I} p_i \cdot \sigma_A^i \otimes \omega_B^i \otimes \gamma_C^i \right\|_1 \leq O\left(|B||C| \cdot \sqrt{\frac{\log |A| + \log |B|}{n} + \frac{\log |A|}{n}}\right)$$

where for linear maps $\mathcal{E}_{A \rightarrow \tilde{A}}$, $\Lambda_{B \rightarrow \tilde{B}}$, and $\Gamma_{C \rightarrow \tilde{C}}$ and operators $\mathcal{X}_{\tilde{A}}$, $\mathcal{Y}_{\tilde{B}}$, and $\mathcal{Z}_{\tilde{C}}$

$$\mathcal{E}_{A \rightarrow \tilde{A}}(\sigma_A^i) = \mathcal{X}_{\tilde{A}}, \quad \Lambda_{B \rightarrow \tilde{B}}(\omega_B^i) = \mathcal{Y}_{\tilde{B}}, \quad \Gamma_{C \rightarrow \tilde{C}}(\gamma_C^i) = \mathcal{Z}_{\tilde{C}} \quad \forall i \in I$$

whenever for the n -shareable extension $\rho_{AB^n C^n}$ we ask that

$$(\mathcal{E}_{A \rightarrow \tilde{A}} \otimes \mathcal{I}_{B^n C^n})(\rho_{AB^n C^n}) = \mathcal{X}_{\tilde{A}} \otimes \rho_{B^n C^n}$$

$$(\Lambda_{B \rightarrow \tilde{B}} \otimes \mathcal{I}_{B^{n-1} C^n})(\rho_{B^n C^n}) = \mathcal{Y}_{\tilde{B}} \otimes \rho_{B^{n-1} C^n}$$

$$(\mathcal{I}_{B^n C^{n-1}} \otimes \Gamma_{C \rightarrow \tilde{C}})(\rho_{B^n C^n}) = \mathcal{Z}_{\tilde{C}} \otimes \rho_{B^n C^{n-1}}.$$

- NB: Standard quantum de Finetti not sufficient for linear constraints [B. et al. 18]

Proof: quantum entropy inequalities

- ▶ Ingredient I: by quantum Pinsker's inequality

$$2\|\rho - \sigma\|_1^2 \leq D(\rho\|\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)]$$

work with **quantum relative entropy** distance [Brandão & Harrow 16/17]

- ▶ Ingredient II: multipartite quantum mutual information [Yang *et al.* 09]

$$I(A_1 : A_2 : \dots : A_n | R)_\rho := \sum_{i=1}^n H(A_i | R) - H(A_1 A_2 \dots A_n | R) - H(R)$$

$$H(R)_\rho := -\text{Tr}[\rho_R \log \rho_R] \quad \text{von Neumann entropy}$$

and corresponding **quantum entropy inequalities**

$$I(A_1 : \dots : A_n | R)_\rho = I(A_1 : A_2 | R)_\rho + I(A_1 A_2 : A_3 | R)_\rho + \dots + I(A_1 \dots A_{n-1} : A_n | R)_\rho$$

$$I(A_1 A_2 : A_3 | R)_\rho = I(A_2 : A_3 | R)_\rho + I(A_1 : A_3 | A_2 R)_\rho$$

- ▶ Ingredient III: measurement $\mathcal{M}_B$ with at most $|B|^6$ outcomes such that

$$\|(\mathcal{I}_A \otimes \mathcal{M}_B)(\rho_{AB} - \sigma_{AB})\|_1 \leq \|\rho_{AB} - \sigma_{AB}\|_1 \leq 2|B| \cdot \|(\mathcal{I}_A \otimes \mathcal{M}_B)(\rho_{AB} - \sigma_{AB})\|_1$$

$\Rightarrow$ optimal distortion relative to quantum side information [Lami *et al.* 18]

Main result: approximation algorithm

Approximation algorithm for free games

We get semi-definite program approximations $\text{sdp}_n(V, \pi, T)$ on the **maximum winning probability** of free games with the guarantee

$$0 \leq \text{sdp}_n(V, \pi, T) - \omega_{Q(T)}(V, \pi) \leq O\left(T^6 \cdot \sqrt{\frac{\log(TA)}{n}}\right)$$

in the form

$$\text{sdp}_n(V, \pi, T) := |T|^2 \max_{\rho} \text{Tr} \left[(V_{A_1 A_2 Q_1 Q_2} \otimes \Phi_{T\hat{T}|S\hat{S}}) \rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})(S\hat{S})} \right]$$

$$\text{s.t. } \rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})^n(S\hat{S})^n} \geq 0, \quad \text{Tr} [\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})^n(S\hat{S})^n}] = 1$$

$$\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})(S\hat{S})} \quad n\text{-shareable on } (A_2 Q_2 \hat{T}) \text{ wrt } (A_1 Q_1 T)(S\hat{S})^n$$

$$\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})(S\hat{S})} \quad n\text{-shareable on } (S\hat{S}) \text{ wrt } (A_1 Q_1 T)(A_2 Q_2 \hat{T})^n$$

$$\text{Tr}_{A_1} [\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})^n(S\hat{S})^n}] = \left(\sum_{q_1} \pi_1(q_1) |q_1\rangle \langle q_1|_{Q_1} \otimes \frac{\text{id}_T}{|T|} \right) \otimes \text{Tr}_{A_1 Q_1 T} [\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})^n(S\hat{S})^n}]$$

$$\text{Tr}_{A_2} [\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})^n(S\hat{S})^n}] = \left(\sum_{q_2} \pi_2(q_2) |q_2\rangle \langle q_2|_{Q_2} \otimes \frac{\text{id}_{\hat{T}}}{|T|} \right) \otimes \text{Tr}_{A_2 Q_2 \hat{T}} [\rho_{(A_1 Q_1 T)(A_2 Q_2 \hat{T})^n(S\hat{S})^n}]$$

- NB: positive partial transpose and dimension agnostic NPA criteria to add

Conclusion

- ▶ Additive ε -approximations on the maximum winning probability $\omega_{Q(T)}(V, \pi)$ of free games with quantum assistance of fixed dimension T with complexity

$$\exp\left(\mathcal{O}\left(\frac{T^{12}}{\varepsilon^2} \log(AT) (\log(Q) + \log(AT))\right)\right)$$

where Q denotes the number of question and A the number of answers

- ▶ In terms of worst case guarantees, this is an **exponential improvement** over previous work [Navascués *et al.* 14/15]
- ▶ Based on novel **multipartite quantum de Finetti** with linear constraints + optimal distortion measurement relative to quantum side information
- ▶ Dependence on T ?
 - ▶ Matching hardness for $T = 1$ classical case [Aaronson *et al.* 14]
 - ▶ Diverges for $T \rightarrow \infty$ consistent with [Ji *et al.* 20]