De Finetti Theorems for Quantum Channels

Mario Berta

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with Borderi, Fawzi, Scholz

QIMP Budapest
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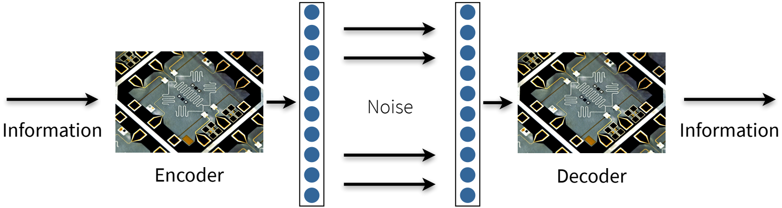
Imperial College London

Outline

Motivation: Noisy Channel Coding

Noisy Channel Coding



Error Correction

m bits are subject to noise modelled by $N(y|x)$, find encoder e and decoder d to maximize probability $p(N, m)$ of retrieving m bits

Noisy Channel Coding (continued)

- Fixed number of bits m and noise model N gives **bilinear optimization**

$$
\begin{aligned}
p(N,m) = \max_{(e,d)} \quad & \frac{1}{2^m} \sum_{x,y,i} N(y|x)d(i|y)e(x|i) \\
\text{s.t.} \quad & \sum_{x} e(x|i) = 1,\ 0 \leq e(x|i) \leq 1 \\
& \sum_{i} d(i|y) = 1,\ 0 \leq d(i|y) \leq 1
\end{aligned}
$$

- Approximating $p(N,m)$ up to multiplicative factor better than $(1 - e^{-1})$ is **NP-hard** in the worst case [Barman & Fawzi 18]

Noisy Channel Coding (continued)

- For the linear program [Hayashi 09, Polyanski *et al.* 10]

$$\begin{aligned} \mathrm{lp}(N,m) = \max_{(r,p)} \quad & \frac{1}{2^m} \sum_{x,y} N(y|x) r_{xy} \\ \text{s.t.} \quad & \sum_x r_{xy} \le 1, \ \sum_x p_x = 2^m \\ & r_{xy} \le p_x, \ 0 \le r_{xy}, p_x \le 1 \end{aligned}$$

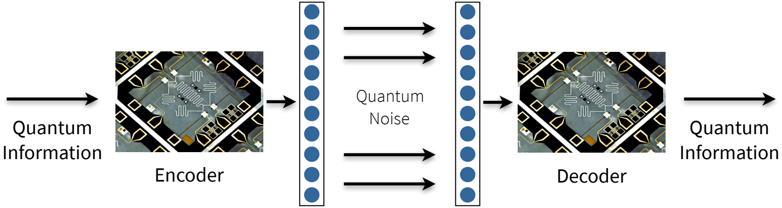
  we have the approximation [Barman & Fawzi 18]

$$p(N,m) \le \mathrm{lp}(N,m) \le \frac{1}{1-e^{-1}} \cdot p(N,m)$$

- Outer bound complemented by **polynomial time** $(1-e^{-1})$-multiplicative approximation algorithms

Quantum Noisy Channel Coding

- Main question: Similar results for quantum error correction? [Matthews 12, Leung & Matthews 15]

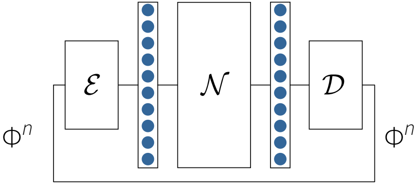


Quantum Error Correction

Find encoder E and decoder D to maximize quantum probability $F(\mathcal{N}, m)$ of retrieving m qubits

Quantum Noisy Channel Coding (continued)

- Near-term quantum devices are of small and intermediate scale

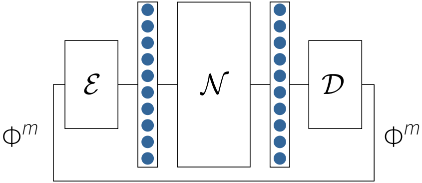


- Tailor-made **approximation algorithms** for encoder/ decoder?

Optimize Quantum Information Processing

Develop mathematical toolbox rooted in optimization theory

Quantum Noisy Channel Coding (continued)



- *m* qubits with quantum noise model $\mathcal{N}$ leads to **quantum channel fidelity**

$$
\begin{aligned}
F(\mathcal{N}, m) := \max \quad & F\Big(\Phi^m, \big((\mathcal{D} \circ \mathcal{N} \circ \mathcal{E}) \otimes \mathcal{I}\big)(\Phi^m)\Big) \\
\text{s.t.} \quad & \mathcal{E}, \mathcal{D} \text{ quantum operations} \\
& \text{(+ physical constraints)}
\end{aligned}
$$

with fidelity $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

Quantum Noisy Channel Coding (continued)

- Via Choi-Jamiołkowski becomes **bilinear optimization**

$$F(\mathcal{N}, m) = \max \quad d_{\bar{A}} d_B \cdot \mathrm{Tr}\left[\left(J^{\mathcal{N}}_{\bar{A}B} \otimes \Phi_{A\bar{B}}\right)\left(E_{A\bar{A}} \otimes D_{B\bar{B}}\right)\right]$$
$$\text{s.t.} \quad E_{A\bar{A}} \geq 0,\ E_A = \frac{1_A}{d_A},\ D_{B\bar{B}} \geq 0,\ D_B = \frac{1_B}{d_B}$$

  with product Choi states $E_{A\bar{A}} \otimes D_{B\bar{B}}$ (where $d_A = d_{\bar{B}} = 2^m$).

- At least as hard as classical problem, approximate solutions?
- **Inner bounds** on figure of merit via from any feasible solution via, e.g.,
  - physical intuition [Bennett *et al.* 96]
  - iterative see-saw methods [Reimpell & Werner 05]
- Main question: **Outer bounds**?

Quantum Noisy Channel Coding (continued)

- Quantum channel fidelity stays the same with free shared randomness assistance

$$
\begin{aligned}
F(\mathcal{N}, n) = \max \quad & d_{\bar{A}} d_B \cdot \mathrm{Tr}\left[\left(J^{\mathcal{N}}_{\bar{A}B} \otimes \Phi_{A\bar{B}}\right)\left(\sum_{i\in I} p_i E^i_{A\bar{A}} \otimes D^i_{B\bar{B}}\right)\right] \\
\text{s.t.} \quad & p_i \geq 0, \ \sum_{i\in I} p_i = 1 \\
& E^i_{A\bar{A}} \geq 0, \ D^i_{B\bar{B}} \geq 0, \ E^i_A = \frac{1_A}{d_A}, \ D^i_B = \frac{1_B}{d_B} \ \forall i \in I
\end{aligned}
$$

  and leads to separable Choi states $\sum_{i\in I} p_i E^i_{A\bar{A}} \otimes D^i_{B\bar{B}}$.

- Idea: Find outer approximations on the set of **separable Choi states** $\mathrm{SEP}_{\mathcal{N}}(A\bar{A}|B\bar{B})$

- NB: Strong hardness for quantum separability problem known [Barak *et al.* 12]

De Finetti Theorems

Monogamous Entanglement

- Quantum states ρ_{AB} are called k-**shareable** if

$$\rho_{AB_1^k} \equiv \rho_{AB_1 \dots B_k} \text{ with } \rho_{AB_j} = \rho_{AB} \ \forall j \in [k]$$

$\Rightarrow$ characterizes separable states [Stoermer 69, Doherty *et al.* 02]

De Finetti for Quantum States

For k-shareable quantum states ρ_{AB} we have that [Christandl *et al.* 07]

$$\left\| \rho_{AB} - \sum_{i \in I} p_i \sigma_A^i \otimes \omega_B^i \right\|_1 \leq \frac{d_B^2}{k}$$

with probabilities $\{p_i\}_{i \in I}$ and quantum states $\sigma_A^i, \omega_B^i \ \forall i \in I$.

- Use $\rho_{AB_1^k} = (\mathcal{I}_A \otimes \pi_{B_1^k})(\rho_{AB_1^k})$ for $\pi_{B_1^k} \in \mathfrak{S}_k$ — the symmetric group of k elements

k-shareable Choi States

De Finetti for Choi States

For states $\rho_{A\bar{A}(B\bar{B})_1^k} = (\mathcal{I}_{A\bar{A}} \otimes \pi_{(B\bar{B})_1^k})(\rho_{A\bar{A}(B\bar{B})_1^k})\ \forall \pi_{(B\bar{B})_1^k} \in \mathfrak{S}_k$ with

$$\rho_{A(B\bar{B})_1^k} = \frac{1_A}{d_A} \otimes \rho_{(B\bar{B})_1^k} \quad \text{and} \quad \rho_{(B\bar{B})_1^{k-1}B_k} = \rho_{(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B}$$

we have that

$$\left\| \rho_{A\bar{A}B\bar{B}} - \sum_{i \in I} p_i \sigma^i_{A\bar{A}} \otimes \omega^i_{B\bar{B}} \right\|_1 \leq \sqrt{\frac{\text{poly}(d)}{k}} \quad \text{for } \sigma^i_A = \frac{1_A}{d_A}, \omega^i_B = \frac{1_B}{d_B}\ \forall i \in I.$$

- Non-signalling: $A \to (B\bar{B})_1^k$ and $B_k \to (B\bar{B})_1^{k-1}$ [Duan & Winter 16]
- Set of **k-shareable Choi states $\mathbf{SH}^k_{\mathcal{N}}(A\bar{A}|B\bar{B})$** characterizes set of separable Choi states $\text{SEP}_{\mathcal{N}}(A\bar{A}|B\bar{B})$

k-shareable Quantum Channels

De Finetti for Quantum Channels

For quantum channels with

$$\mathcal{N}_{AB_1^k\to\bar{A}\bar{B}_1^k}\left(\left(\mathcal{I}_A\otimes\pi_{B_1^k}\right)(\cdot)\right)=\left(\mathcal{I}_A\otimes\pi_{\bar{B}_1^k}\right)\left(\mathcal{N}_{AB_1^k\to\bar{A}\bar{B}_1^k}(\cdot)\right)\quad\forall\pi_{B_1^k},\pi_{\bar{B}_1^k}\in\mathfrak{S}_k$$

$$\mathrm{Tr}_{\bar{A}}\left[\mathcal{N}_{AB_1^k\to\bar{A}\bar{B}_1^k}(\cdot)\right]=\mathrm{Tr}_{\bar{A}}\left[\mathcal{N}_{AB_1^k\to\bar{A}\bar{B}_1^k}\left(\frac{1_A}{d_A}\otimes\mathrm{Tr}_A[\cdot]\right)\right]$$

$$\mathrm{Tr}_{\bar{B}_k}\left[\mathcal{N}_{AB_1^k\to\bar{A}\bar{B}_1^k}(\cdot)\right]=\mathrm{Tr}_{\bar{B}_k}\left[\mathcal{N}_{AB_1^k\to\bar{A}\bar{B}_1^k}\left(\mathrm{Tr}_{B_k}[\cdot]\otimes\frac{1_{B_k}}{d_B}\right)\right]$$

we have that

$$\left\|\mathcal{N}_{AB\to\bar{A}\bar{B}}-\sum_{i\in I}p_i\mathcal{E}^i_{A\to\bar{A}}\otimes\mathcal{D}^i_{B\to\bar{B}}\right\|_\diamond\leq\sqrt{\frac{\mathrm{poly}(d)}{k}}$$

with probabilities $\{p_i\}_{i\in I}$ and $\mathcal{E}^i_{A\to\bar{A}},\mathcal{D}^i_{B\to\bar{B}}$ quantum channels $\forall i\in I$.

Application: Noisy Channel Coding

Quantum Channel Fidelity

- Shared randomness assisted version

$$
\begin{aligned}
F(\mathcal{N}, m) = \max \quad & d_{\bar{A}} d_B \cdot \mathrm{Tr}\left[\left(J^{\mathcal{N}}_{\bar{A}B} \otimes \Phi_{A\bar{B}}\right)\left(\sum_{i \in I} p_i E^i_{A\bar{A}} \otimes D^i_{B\bar{B}}\right)\right] \\
\text{s.t.} \quad & p_i \geq 0, \ \sum_{i \in I} p_i = 1 \\
& E^i_{A\bar{A}} \geq 0, \ D^i_{B\bar{B}} \geq 0, \ E^i_A = \frac{1_A}{d_A}, \ D^i_B = \frac{1_B}{d_B} \ \forall i \in I
\end{aligned}
$$

- Idea: Approximate separable Choi states $\sum_{i \in I} p_i E^i_{A\bar{A}} \otimes D^i_{B\bar{B}}$ by k-shareable Choi states $W_{A\bar{A}(B\bar{B})^k_1}$
- k-shareable has **semi-definite representation**

Outer Bound Approximations

- **Efficiently computable** semi-definite program outer bounds

$$\mathrm{sdp}_k(\mathcal{N}, m) := \max \quad d_{\bar{A}} d_B \cdot \mathrm{Tr}\left[\left(\mathcal{N}_{\bar{A}\to B_1}\left(\Phi_{\bar{A}\bar{A}}\right) \otimes \Phi_{A\bar{B}_1}\right) W_{A\bar{A}B_1\bar{B}_1}\right]$$

$$\text{s.t.} \quad W_{A\bar{A}(B\bar{B})_1^k} \geq 0,\ \mathrm{Tr}\left[W_{A\bar{A}(B\bar{B})_1^k}\right] = 1,\ W_{A\bar{A}(B\bar{B})_1^k} = \left(\mathcal{I}_{A\bar{A}} \otimes \pi_{(B\bar{B})_1^k}\right)\left(W_{A\bar{A}(B\bar{B})_1^k}\right)$$

$$W_{A(B\bar{B})_1^k} = \frac{1_A}{2^m} \otimes W_{(B\bar{B})_1^k},\ W_{A\bar{A}(B\bar{B})_1^{k-1}B_k} = W_{A\bar{A}(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B}$$

$$\mathrm{PPT}\left(A\bar{A} : (B\bar{B})_1^k\right) \geq 0 \quad \text{(positive partial transpose)}$$

- With **approximation guarantee** to quantum channel fidelity

$$|\mathrm{spd}_k(\mathcal{N}, m) - F(\mathcal{N}, m)| \leq \sqrt{\frac{\mathrm{poly}(d)}{k}}$$

- Previous: [Matthews 12, Leung & Matthews 15, Tomamichel *et al.* 16, Wang *et al.* 16/17] and [Rozpedek *et al.* 18, Kaur *et al.* 18]

Certifying Optimality of Relaxations

- Compare classical linear program relaxation [Barman & Fawzi 18]

$$p(N,m) \leq \mathrm{lp}(N,m) \leq \frac{1}{1-e^{-1}} \cdot p(N,m)$$

- No finite approximation guarantee for $F(\mathcal{N},m) \leq \mathrm{sdp}_k(\mathcal{N},m)$

Rank Loop Conditions

If for $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that

$$\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^k}\right) \leq \max\left\{\mathrm{rank}\left(W_{A\bar{A}(B\bar{B})_1^l}\right), \mathrm{rank}\left(W_{(B\bar{B})_1^{k-l}}\right)\right\}$$

then we have equality $\mathrm{sdp}_k(\mathcal{N},m) = F(\mathcal{N},m)$

- Proof via [Navascués *et al.* 09]

Numerical Example Relaxations

- Uniform noise corresponds to **qubit depolarizing channel**

$$\mathrm{Dep}_p : \rho \mapsto p \cdot \frac{1_B}{2} + (1-p) \cdot \rho \quad \text{with } p \in [0, 4/3].$$

Question

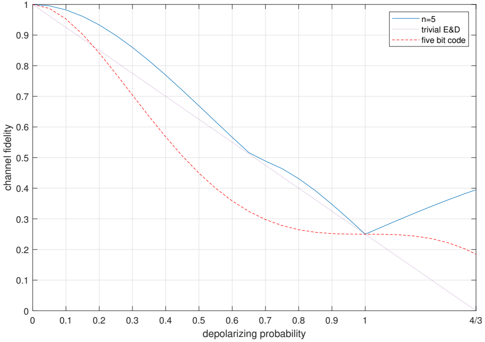
What is the optimal code for reliably storing $m = 1$ qubit in noisy 5 qubit quantum memory, that is, $p\left(\mathrm{Dep}_p^{\otimes 5}, 1\right) = ?$

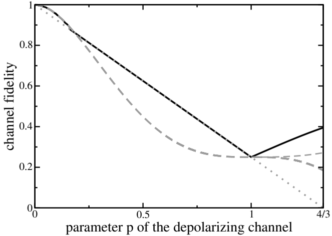
- Analytical [Bennett *et al.* 96] as well as see-saw [Reimpell & Werner 05] lower bounds, our work upper bounds

$$p\left(\mathrm{Dep}_p^{\otimes 5}, 1\right) \leq \mathrm{sdp}_k\left(\mathrm{Dep}_p^{\otimes 5}, 1\right)$$

Numerical Example Relaxations (continued)

- Exploiting symmetries for analytical **dimension reduction** for first level $\mathrm{sdp}_1\left(\mathrm{Dep}_p^{\otimes 5}, 1\right)$ [Wang *et al.* 16/17] gives





[Reimpell & Werner 05] lower bounds

- $p \in [1, 4/3]$ [Reimpell & Werner 05] optimal, $p \in [0, 0.18]$ room for improved codes

Conclusion

Conclusion

- Quantum noisy channel coding (one-shot) via **de Finetti theorem for quantum channels**
- Optimization theory tools to numerically study quantum error correction for practical settings
- Variations possible, e.g., classical communication assistance, physical constraints

Open Questions

- Numerics via dimension reduction? Polynomial size + symdpoly numerics [Rosset]?
- Settings with **provably good quantum meta-converse**?
- Optimal quantum de Finetti: dimension dependence, minimal conditions?

Proofs Ideas (board)

Proof Ideas: De Finetti for Choi States

Let $\rho_{A\bar{A}(B\bar{B})_1^k}$ be quantum states such that for $\pi \in \mathfrak{S}_k$

$$\rho_{A\bar{A}(B\bar{B})_1^k} = (\mathcal{I}_{A\bar{A}} \otimes \pi_{(B\bar{B})_1^k})(\rho_{A\bar{A}(B\bar{B})_1^k})$$

$$\rho_{A(B\bar{B})_1^k} = \frac{1_A}{d_A} \otimes \rho_{(B\bar{B})_1^k}$$

$$\rho_{(B\bar{B})_1^{k-1}B_k} = \rho_{(B\bar{B})_1^{k-1}} \otimes \frac{1_{B_k}}{d_B}\,.$$

Then, we have that

$$\left\| \rho_{A\bar{A}B\bar{B}} - \sum_{i\in I} p_i \sigma^i_{A\bar{A}} \otimes \omega^i_{B\bar{B}} \right\|_1 \leq \sqrt{\frac{\mathrm{poly}(d)}{k}}$$

with probabilities $\{p_i\}_{i\in I}$ and $\sigma^i_A = \frac{1_A}{d_A}$, $\omega^i_B = \frac{1_B}{d_B}$ $\forall i \in I$.

Proof Ideas: De Finetti with Linear Constraints

Let $\rho_{AB_1^k}$ be quantum states, $\Lambda_{A\to C_A}, \Gamma_{B\to C_B}$ linear maps, and X_{C_A}, Y_{C_B} operators such that for $\pi \in \mathfrak{S}_k$

$$\left(\mathcal{I}_A \otimes \pi_{B_1^k}\right)(\rho_{AB_1^k}) = \rho_{AB_1^k} \qquad k\text{-shareable on } B$$

$$\left(\Lambda_{A\to C_A} \otimes \mathcal{I}_{B_1^k}\right)(\rho_{AB_1^k}) = X_{C_A} \otimes \rho_{B_1^k} \qquad \text{linear constraint on } A$$

$$\Gamma_{B_k\to C_B}(\rho_{B_1^n}) = \rho_{B_1^{k-1}} \otimes Y_{C_B} \qquad \text{linear constraint on } B.$$

Then, we have that

$$\left\|\rho_{AB} - \sum_{i\in I} p_i \sigma_A^i \otimes \omega_B^i\right\|_1 \leq \sqrt{\frac{d_B^4 (d_B+1)^2 \log d_A}{k}}$$

with probabilities $\{p_i\}_{i\in I}$ and quantum states σ_A^i, ω_B^i such that $\forall i \in I$

$$\Lambda_{A\to C_A}\left(\sigma_A^i\right) = X_{C_A} \quad \text{and} \quad \Gamma_{B\to C_B}\left(\omega_B^i\right) = Y_{C_B}.$$

Application: Bilinear Optimization

De Finetti with linear constraints gives **outer hierarchy for programs of the bilinear form**

$$\begin{aligned} \max \quad & \mathrm{Tr}\big[H(D \otimes E)\big] \\ \text{s.t.} \quad & D \in \mathcal{S}_D, E \in \mathcal{S}_E \end{aligned}$$

where H is a matrix and $\mathcal{S}_D$ and $\mathcal{S}_E$ are positive semi-definite representable sets of the form

$$\mathcal{S}_D = \Pi_{A \to D}(\mathcal{S}_A^+ \cap \mathcal{A}_A) \quad \text{and} \quad \mathcal{S}_E = \Pi_{B \to E}(\mathcal{S}_B^+ \cap \mathcal{A}_B)$$

with $\Pi_{A \to D}, \Pi_{B \to E}$ linear maps, $\mathcal{S}_A^+, \mathcal{S}_B^+$ the set of density operators, and $\mathcal{A}_A, \mathcal{A}_B$ affine subspaces of matrices.