The Quantum Reverse Shannon Theorem and other Channel Simulations

Mario Berta (Fernando Brandão, Matthias Christandl, Renato Renner, Stephanie Wehner)

Quantum Reverse Shannon Theorem

- Previously proved by Bennett, Devetak, Harrow, Shor and Winter [1].
- New proof based on one-shot Quantum State Merging [2,3] and the Post-Selection Technique for Quantum Channels [4].
- Outline:
    - Understanding the Theorem (Classical and Quantum Shannon Theory)
    - Idea of our Proof
    - Quantum State Merging
    - Post-Selection Technique
    - Other Channel Simulations

[1] arXiv.org/quant-ph:0912.5537
[2] Horodecki et al., Nature 436:673-676, 2005
[3] Berta, arXiv.org/quant-ph:0912.4495
[4] Christandl et al., Phys. Rev. Lett. 102:020504, 2009

Shannon's Classical Noisy Channel Coding Theorem

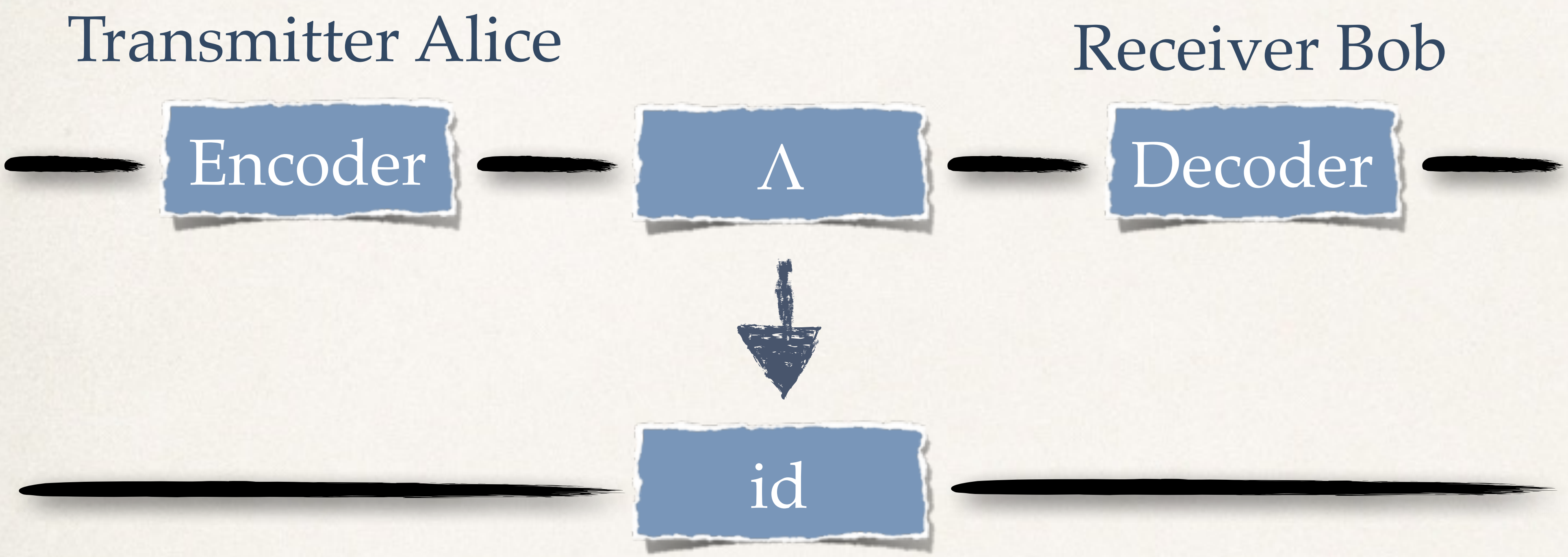
Transmitter Alice

Receiver Bob

Λ: noisy channel

How many bits can Alice transmit on average per use of the channel?

Shannon's Classical Noisy Channel Coding Theorem



Λ: noisy channel
id: perfect channel

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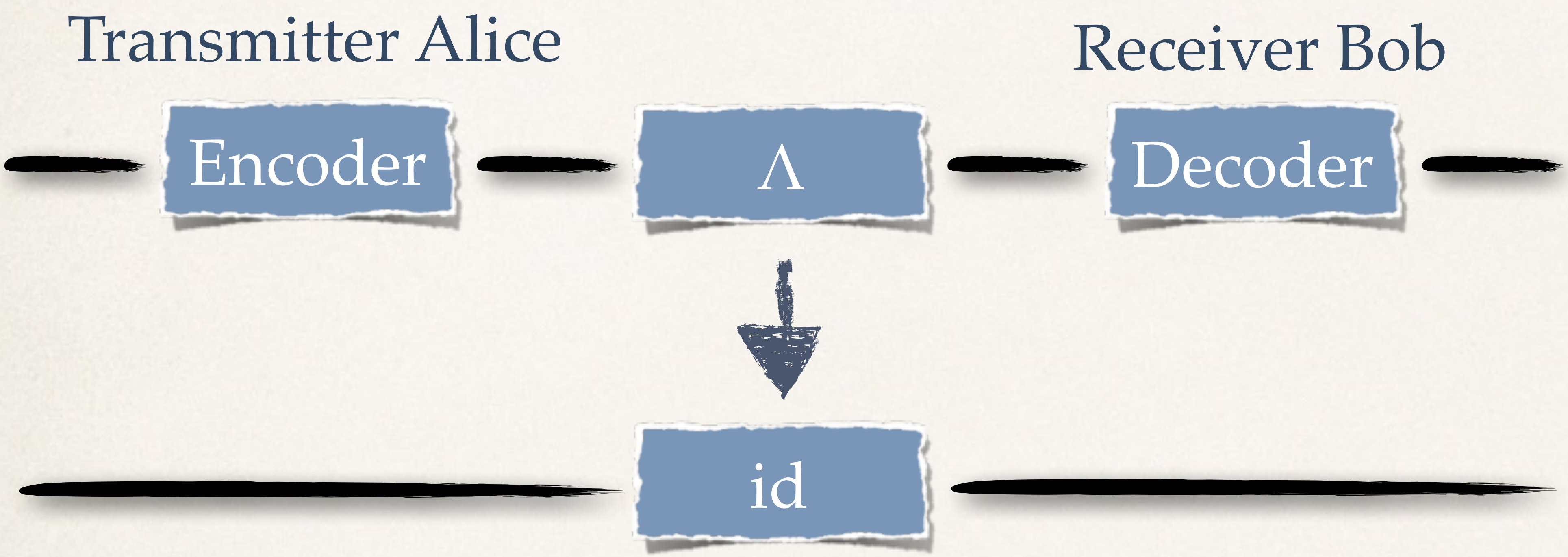

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How many bits can Alice transmit on average per use of the channel?

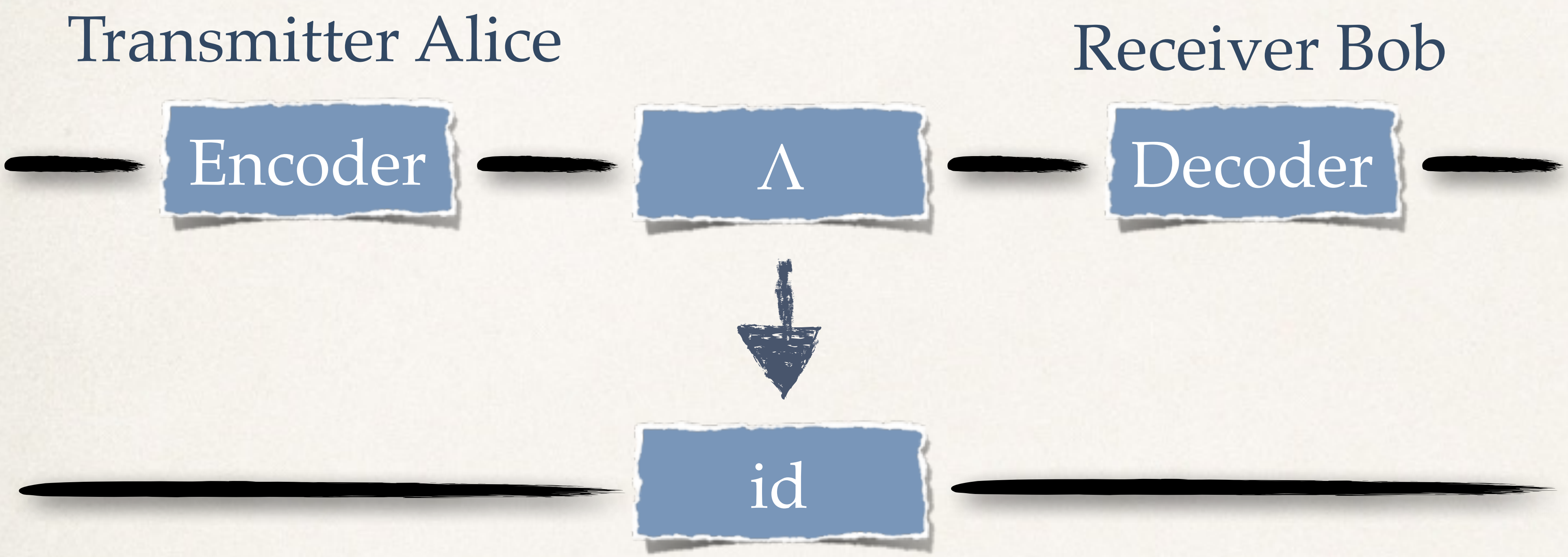
⇒ Asymptotic channel capacity [5]:

$$C(\Lambda) = \max_X (H(X) + H(\Lambda(X)) - H(X, \Lambda(X)))$$

$$H(X) = -\sum_x p_x \log p_x$$

[5] Shannon, Bell. Syst. Tech. J. 27:379-423,623-656, 1948

Shannon's Classical Noisy Channel Coding Theorem



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Note: Neither back communication nor shared randomness help

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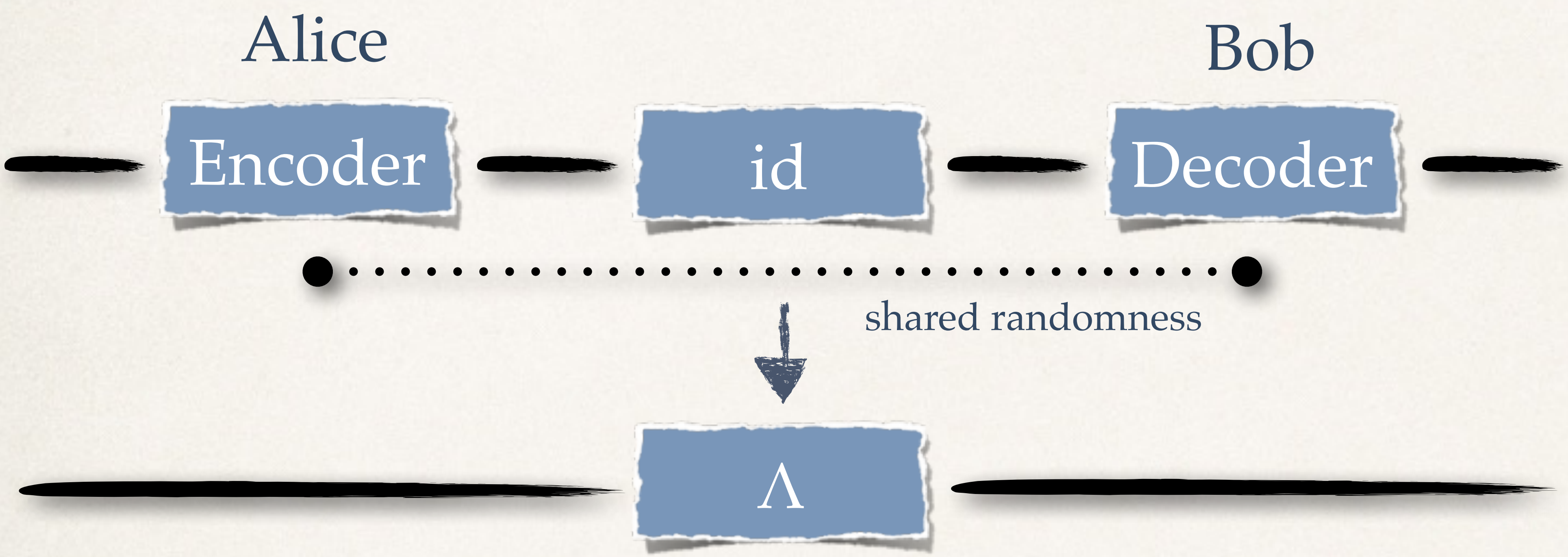
Classical Reverse Shannon Theorem



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Using shared randomness, at what asymptotic rate can the id-channel simulate a channel Λ?

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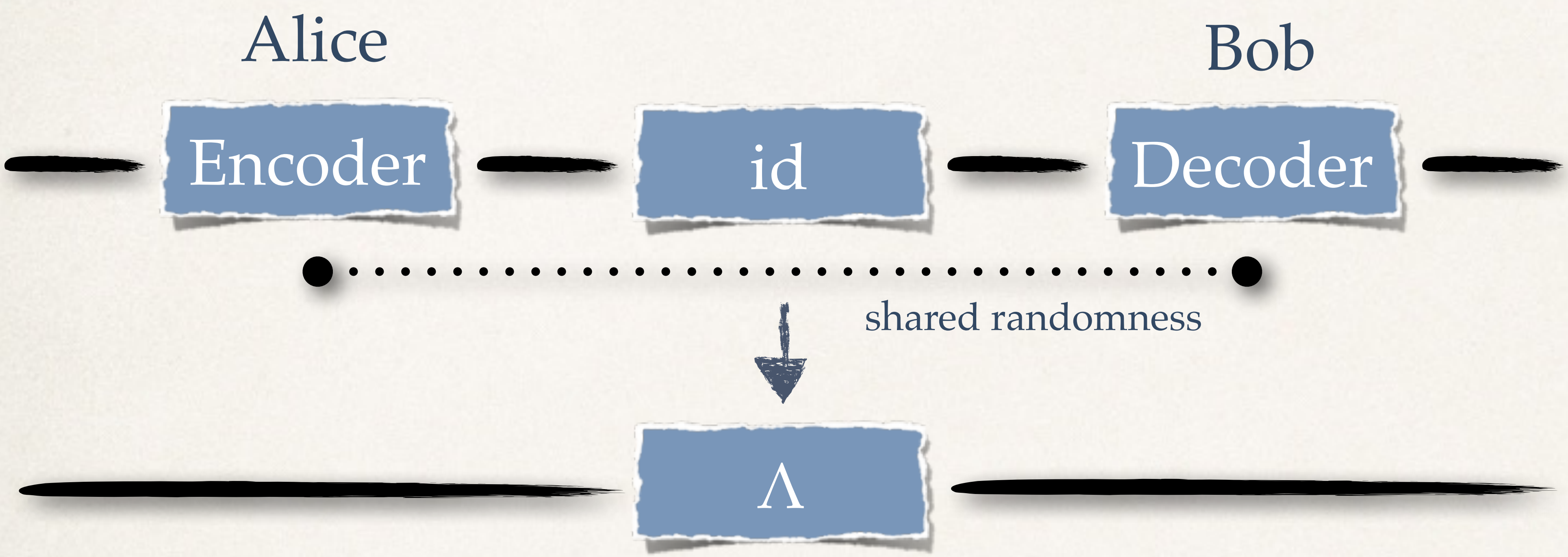

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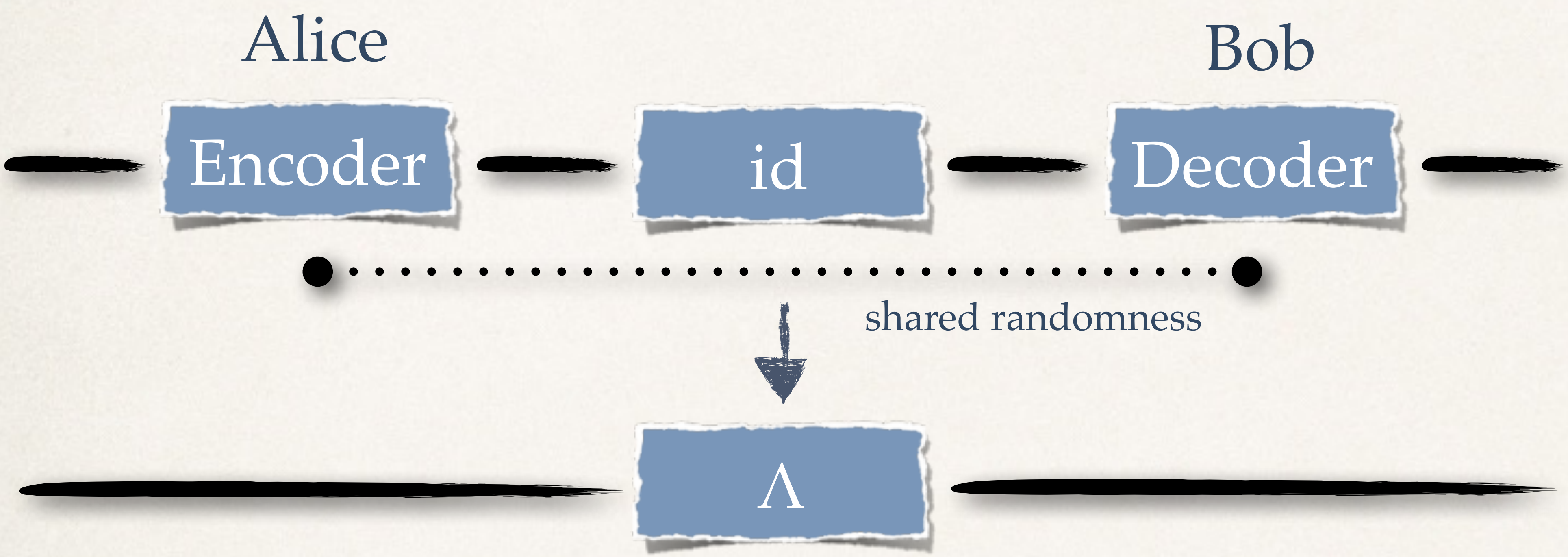

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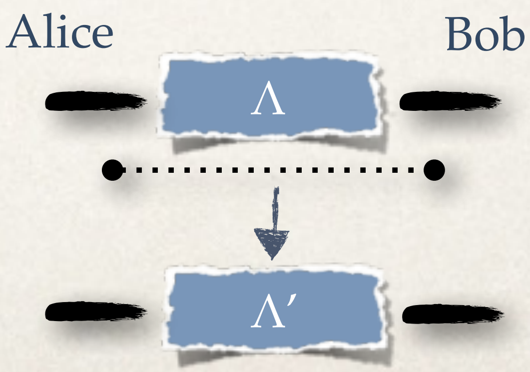
⇒ C(Λ) as well [6]! I.e. the asymptotic capacity of a channel Λ to simulate another channel Λ′ in the presence of free shared randomness is given by:

$$C_R(\Lambda, \Lambda') = \frac{C(\Lambda)}{C(\Lambda')}$$



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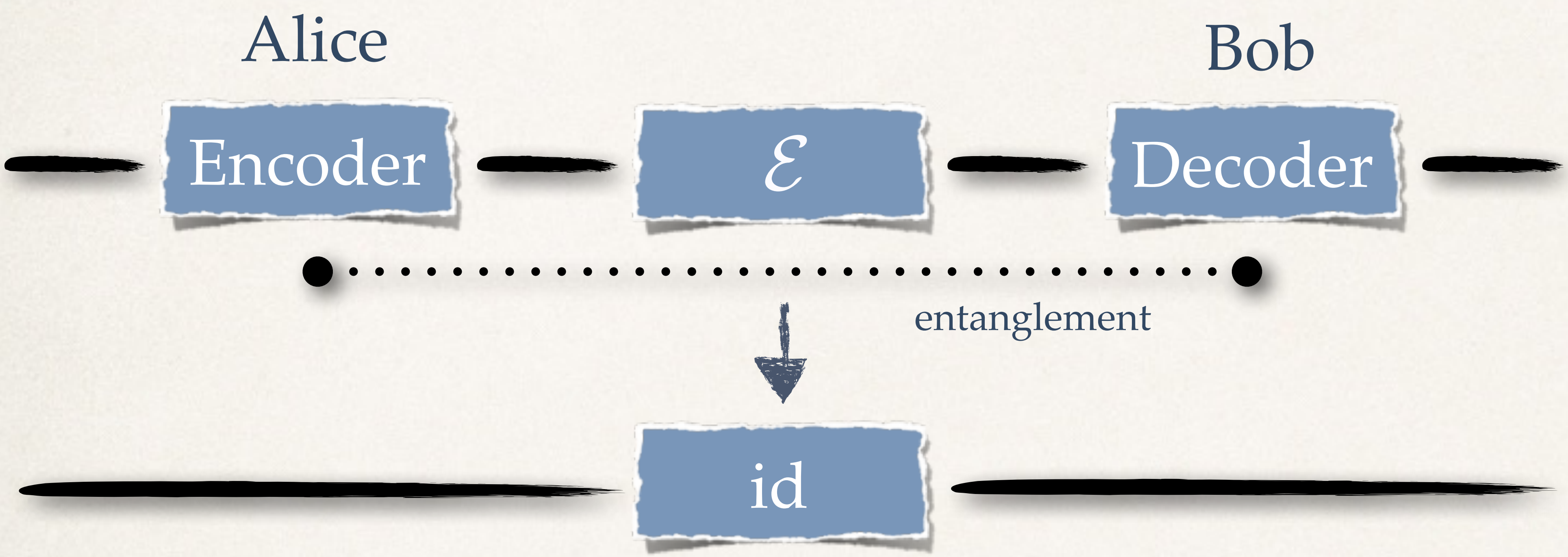
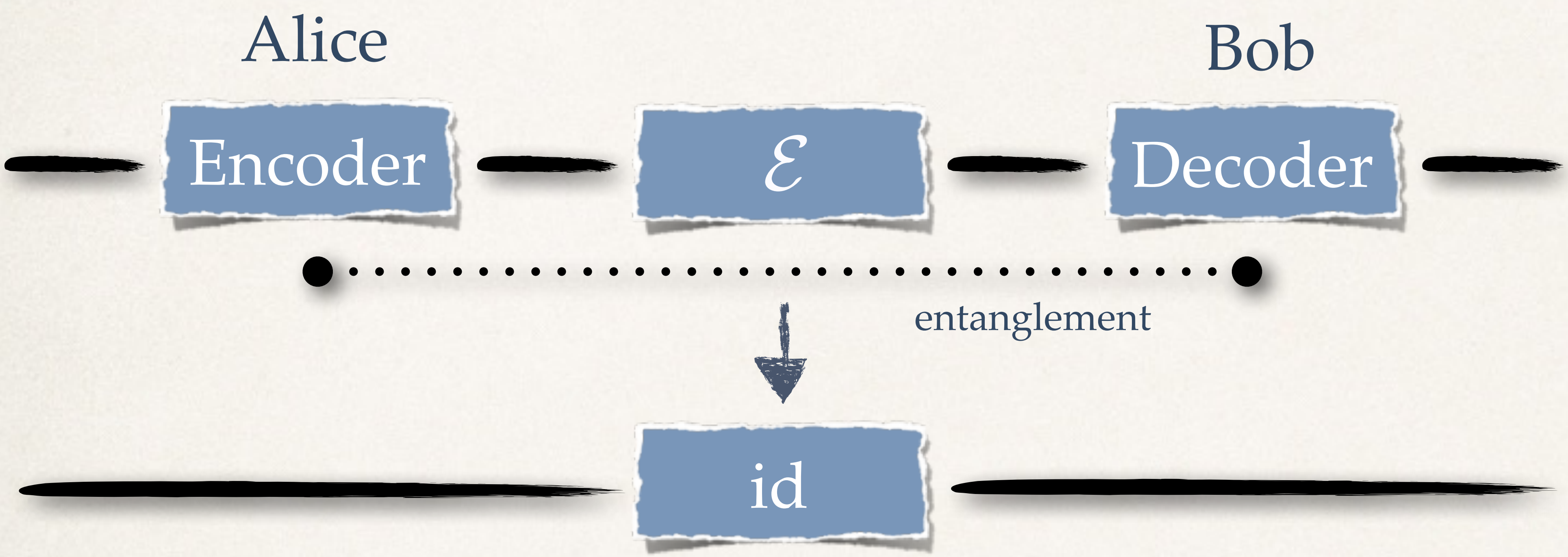
Quantum Shannon Theorem



$\mathcal{E}$: quantum channel
id: perfect classical channel

Using entanglement, at what asymptotic rate can Alice transmit classical information?

Quantum Shannon Theorem



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⇒ Asymptotic entanglement-assisted classical capacity [6]:

$$C_E = \max_{\rho}(H(\rho) + H(\mathcal{E}(\rho)) - H((\mathcal{E} \otimes \mathrm{id})\Phi_{\rho}))$$

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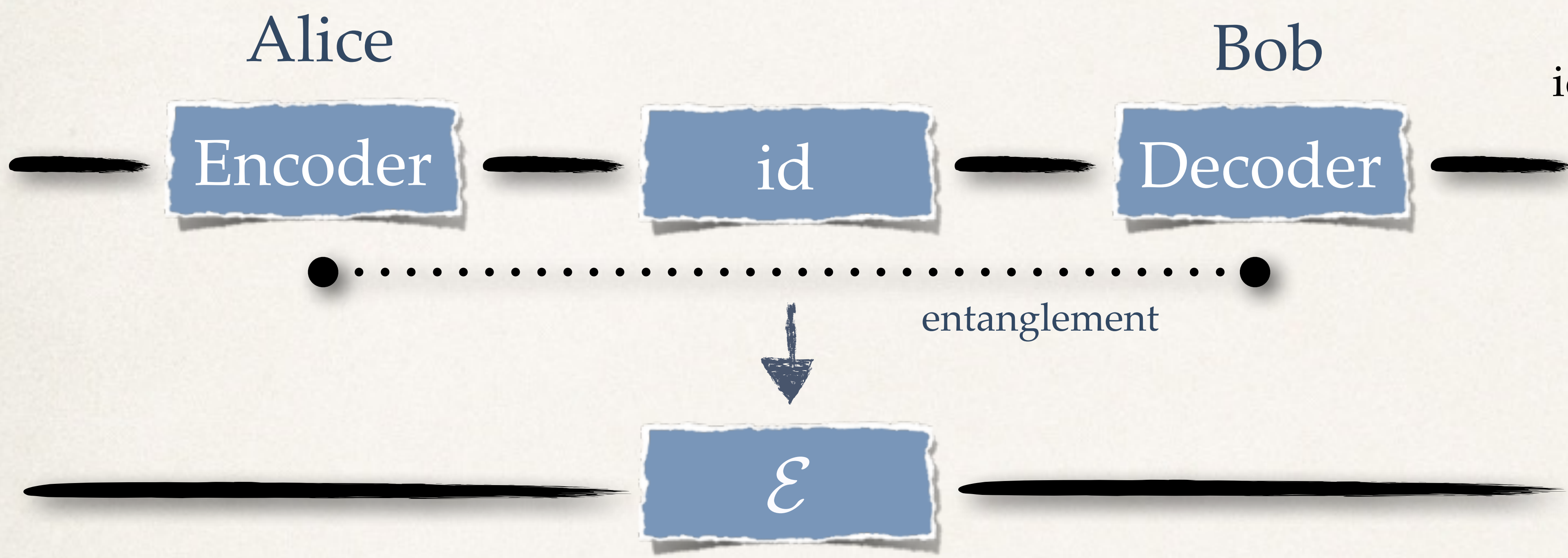

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$\mathcal{E}$

Using entanglement, at what asymptotic rate can the classical id-channel simulate a quantum channel?

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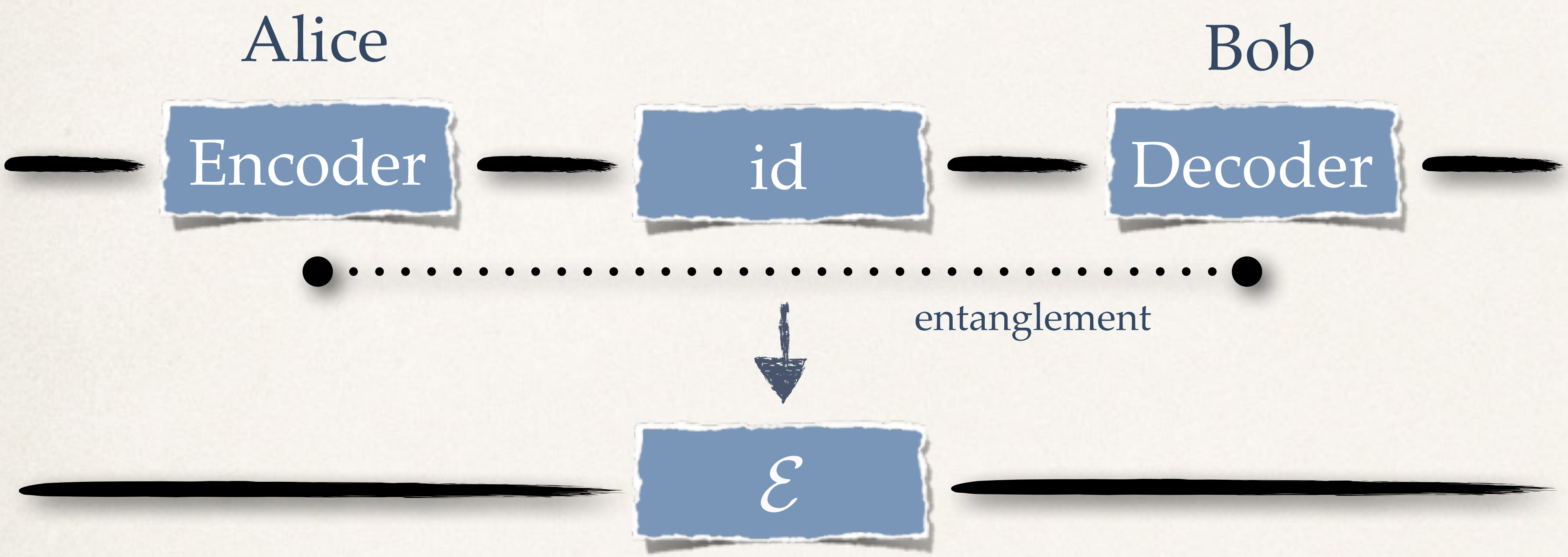

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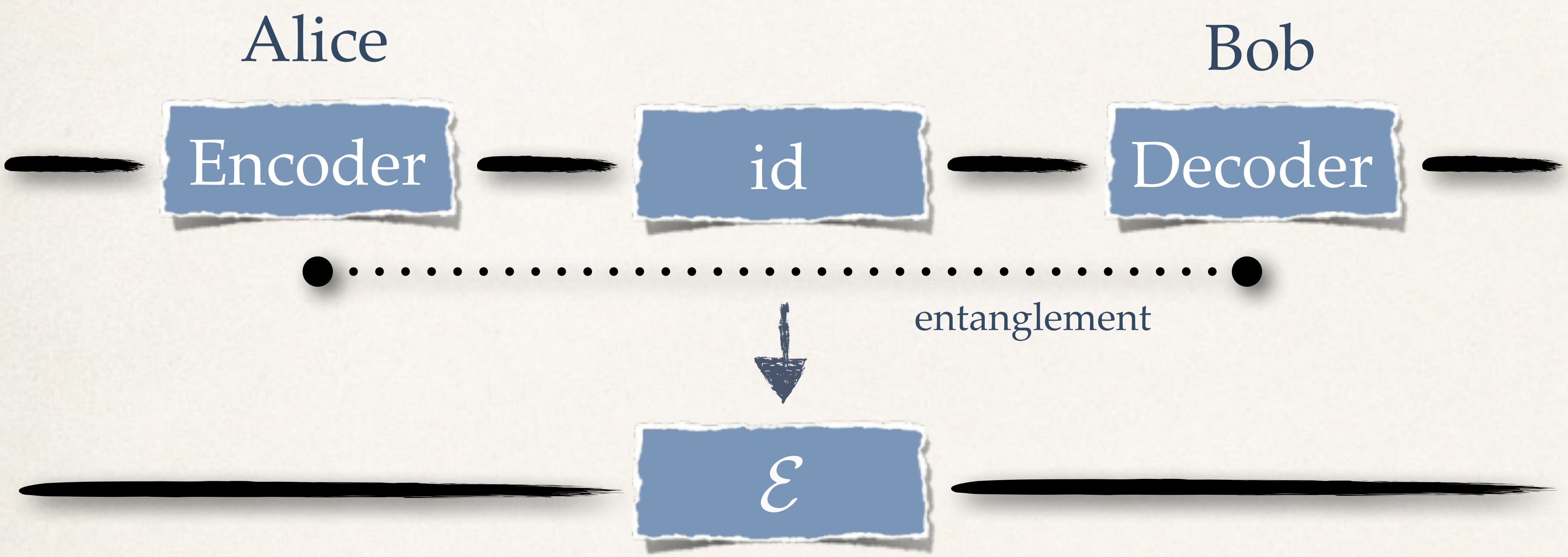

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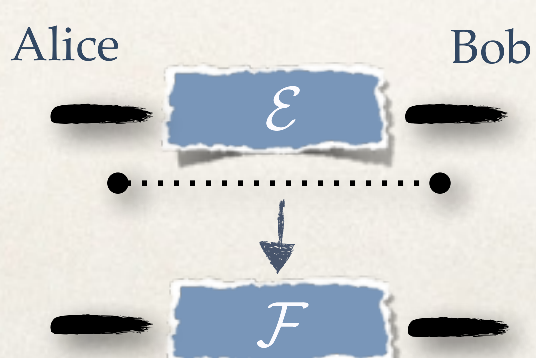
Using entanglement, at what asymptotic rate can the classical id-channel simulate a quantum channel?
⇒ C_E as well! I.e. the asymptotic capacity of a quantum channel to simulate another quantum channel in the presence of free entanglement is given by:

$$C_E(\mathcal{E},\mathcal{F}) = \frac{C_E(\mathcal{E})}{C_E(\mathcal{F})}$$



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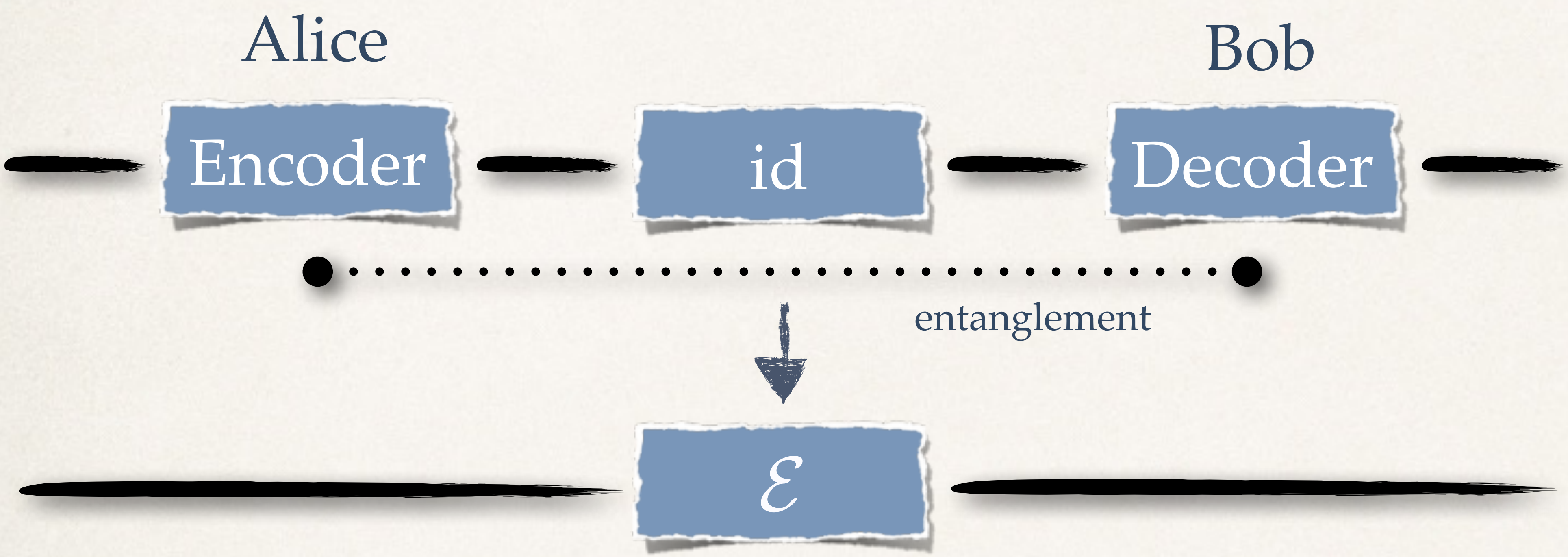

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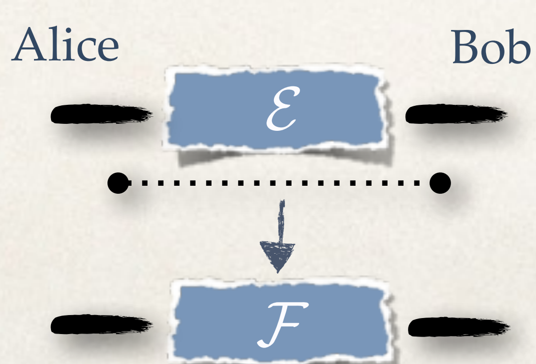

Note: Maximally entangled states are not sufficient, embezzling states needed!

Embezzling States

- Introduced by Van Dam and Hayden [7]
- **Definition:** A pure, bipartite state of the form

$$|\mu(k)\rangle_{AB} = \frac{1}{\sqrt{G(k)}} \sum_{j=1}^{k} \frac{1}{\sqrt{j}} |jj\rangle_{AB}$$

where $G(k) = \sum_{j=1}^{k} \frac{1}{j}$, is called *embezzling state* of index *k*.

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- **Proposition:** Let $\epsilon > 0$ and let $|\varphi\rangle_{AB}$ be a pure bipartite state of Schmidt rank *m*. Then the transformation

$$|\mu(k)\rangle_{AB} \mapsto |\mu(k)\rangle_{AB} \otimes |\varphi\rangle_{AB}$$

  can be accomplished with fidelity better than $(1-\epsilon)$ for $k > m^{1/\epsilon}$ with local isometries at *A* and *B*.
- **Definition:** The *fidelity* between two density matrices ϱ and σ is defined as

$$F(\rho, \sigma) = (\mathrm{tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}))^2$$

  and it is a notion of distance on the set of density matrices.

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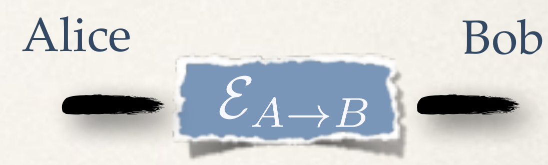
Our Proof

- $\mathcal{E}_{A\to B}$ CPTP map to simulate, $\mathcal{E}_{A\to B}: \quad S(\mathcal{H}_A) \to S(\mathcal{H}_B)$
  $\rho_A \mapsto \mathcal{E}_{A\to B}(\rho_A)$



- **Stinespring Dilation:**

  $$\mathcal{E}_{A\to B}(\rho_A) = \mathrm{tr}_{A'}(U_{A\to BA'}\rho_A U_{A\to BA'}) =: \mathrm{tr}_{A'}(\sigma_{BA'})$$

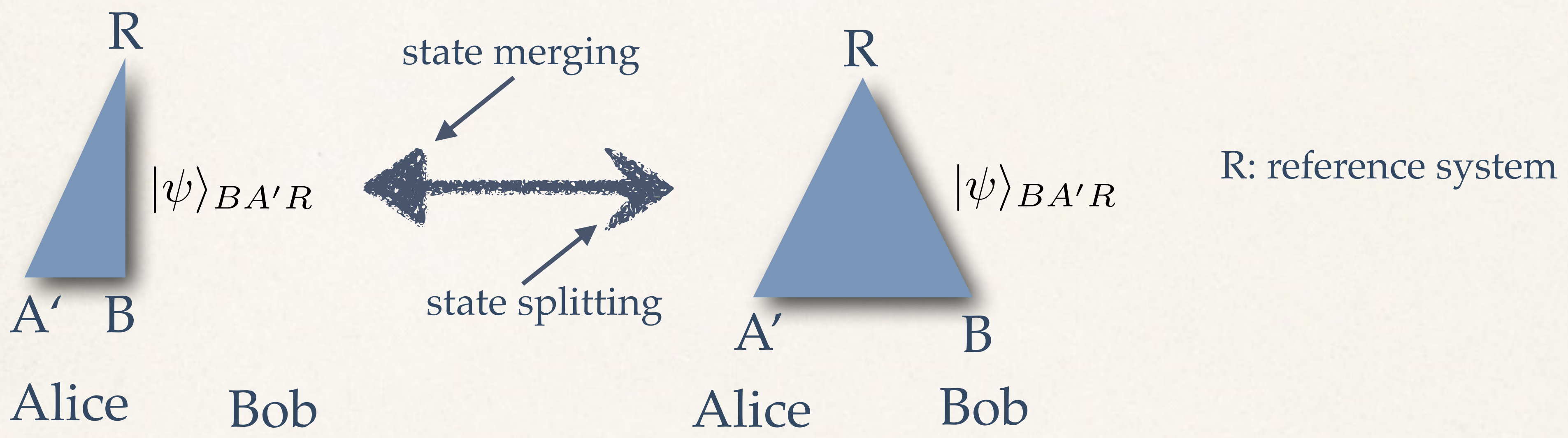
  for some isometry $U_{A\to BA'} : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{A'}$, with $\dim(\mathcal{H}_{A'}) \leq \dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$.

- **Key Idea:**
  (i) Local simulation of $\mathcal{E}_{A\to B}$ at Alice's side using Stinespring Dilation
  $\Rightarrow$ $\sigma_{BA'}$ at Alice's side.
  (ii) Send part B of $\sigma_{BA'}$ to Bob with classical channel and entanglement
  $\Rightarrow$ Bob has $\sigma_B = \mathcal{E}_{A\to B}(\rho_A)$!

Quantum State Merging/State Splitting

R: reference system

Quantum State Merging/State Splitting



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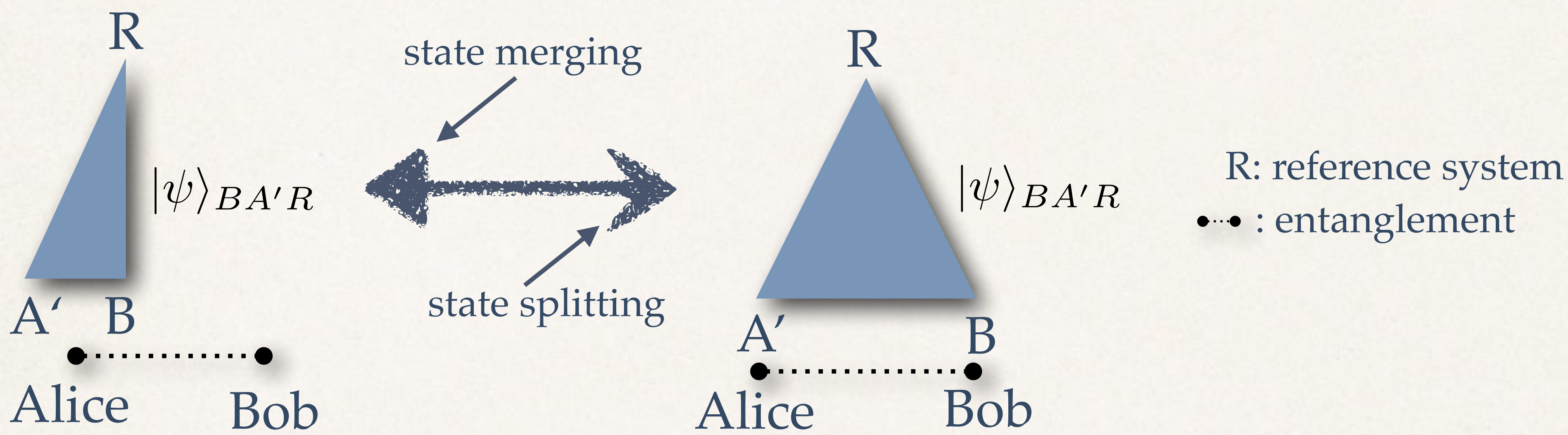
Quantum State Merging/State Splitting



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- Our case: $\sigma_{BA'} \to \sigma_{BA'R} = |\psi\rangle\langle\psi|_{BA'R}$ purification, free entanglement, classical communication to quantify.

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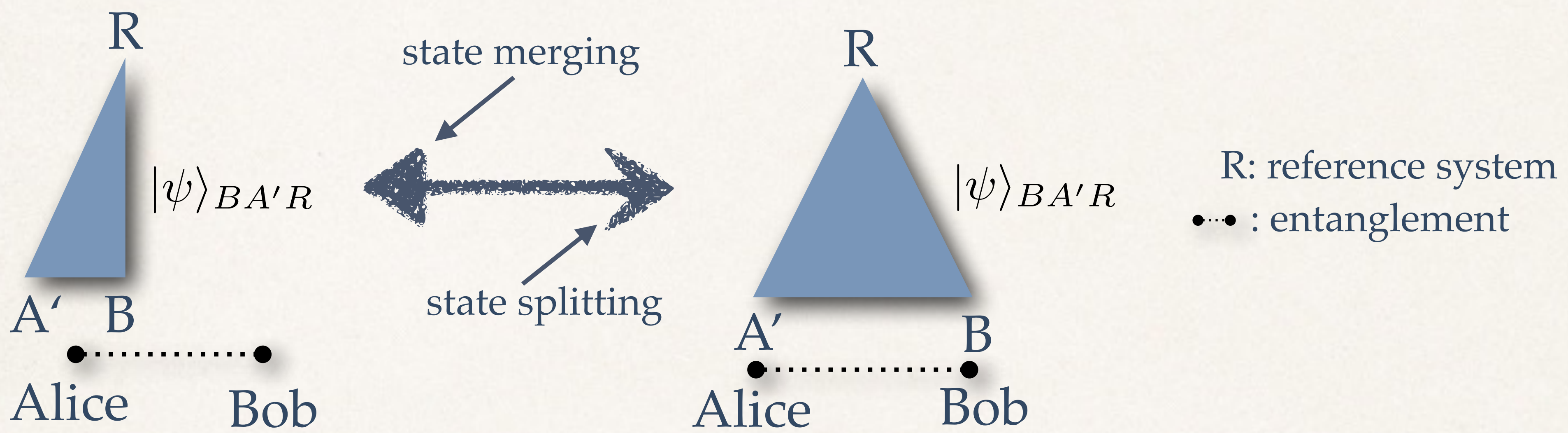

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- Horodecki et al. [2], $|\psi^{\otimes n}\rangle_{BA'R}$ with classical communication cost c_n :

$$c = \lim_{n\to\infty} \frac{1}{n} c_n = H(\sigma_B) + H(\sigma_R) - H(\sigma_{BR}) = I(B:R)_\sigma$$

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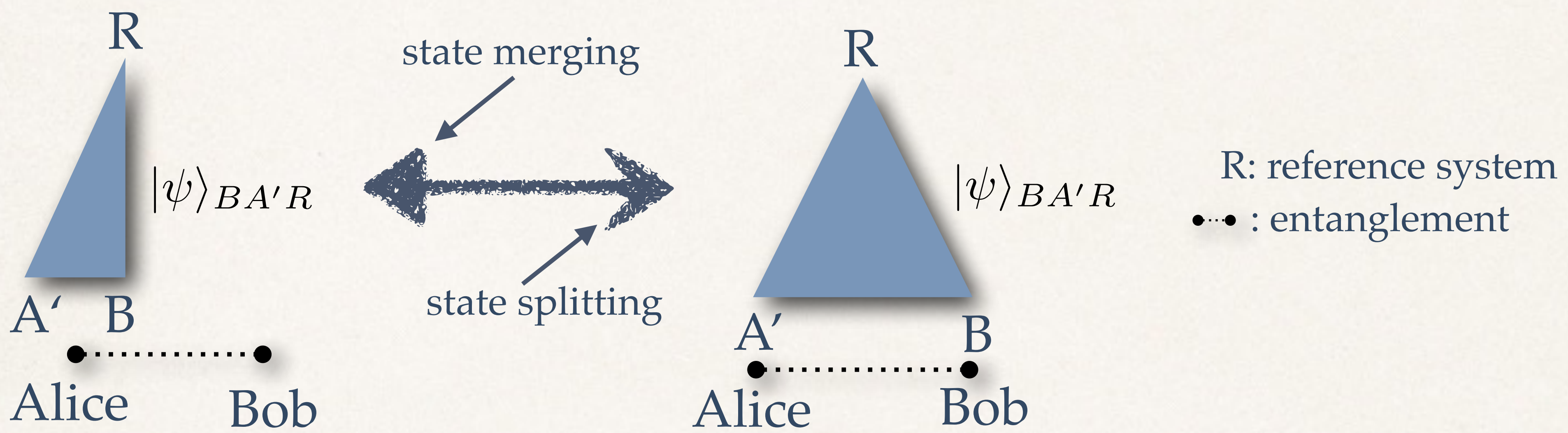

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- One-shot version, $|\psi\rangle_{BA'R}$ with classical communication cost c_ϵ for an error ϵ :

$$c_\epsilon \cong I^\epsilon_{\max}(B:R)_\sigma$$

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Back to the Proof

- CPTP map $\mathcal{E}^{\otimes n}_{A\to B}(\rho^n_A) = \mathrm{tr}_{A'}(U^n_{A\to BA'}\rho^n_A U^n_{A\to BA'}) =: \mathrm{tr}_{A'}(\sigma^n_{BA'})$ to simulate.
- Local simulation of $U^n_{A\to BA'}$ and state splitting of $\sigma^n_{BA'}$ gives ε-approximation $\mathcal{F}^{n,\epsilon}_{A\to B}$ of $\mathcal{E}^{\otimes n}_{A\to B}$ for a class. comm. cost $I^{\epsilon}_{\max}(B:R)_{\sigma^n}$.

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- **Definition:** Let $\mathcal{E}$ be a quantum operation. The *diamond norm* [8] of $\mathcal{E}$ is defined as
$$\|\mathcal{E}\|_\diamond = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1\leq 1} \|(\mathcal{E}\otimes \mathrm{id}_k)(\sigma)\|_1$$
$$\|\sigma\|_1 = \mathrm{tr}(\sqrt{\sigma^\dagger\sigma})\,.$$
The induced metric is a notion of distance for quantum operations.

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- To show: $\lim_{\epsilon\to 0}\lim_{n\to\infty} \|\mathcal{E}^{\otimes n}_{A\to B} - \mathcal{F}^{n,\epsilon}_{A\to B}\|_\diamond = 0$, $\lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I^{\epsilon}_{\max}(B:R)_{\sigma^n} = C_E$.

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The Post-Selection Technique

- Christandl et al. [4]: Let $\mathcal{E}_{A^n}$ and $\mathcal{F}_{A^n}$ be quantum operations that act permutation-covariant on a *n*-partite system $\mathcal{H}_{A^n} = \mathcal{H}_A^{\otimes n}$. Then

$$\|\mathcal{E}_{A^n} - \mathcal{F}_{A^n}\|_\diamond \leq \mathrm{poly}(n)\|((\mathcal{E}_{A^n} - \mathcal{F}_{A^n}) \otimes \mathrm{id}_{R^n R'})(\zeta_{A^n R^n R'})\|_1$$

where $\zeta_{A^n R^n R'}$ is a purification of the (de Finetti type) state

$$\zeta_{A^n R^n} = \int \omega_{AR}^{\otimes n} d(\omega_{AR})$$

with ω_{AR} a pure state on $\mathcal{H}_A \otimes \mathcal{H}_R, \mathcal{H}_R \cong \mathcal{H}_A, \mathcal{H}_{R^n} = \mathcal{H}_R^{\otimes n}$ and *d(.)* the measure on the normalized pure states on $\mathcal{H}_A \otimes \mathcal{H}_R$ induced by the Haar measure on the unitary group acting on $\mathcal{H}_A \otimes \mathcal{H}_R$, normalized to $\int d(.) = 1$.

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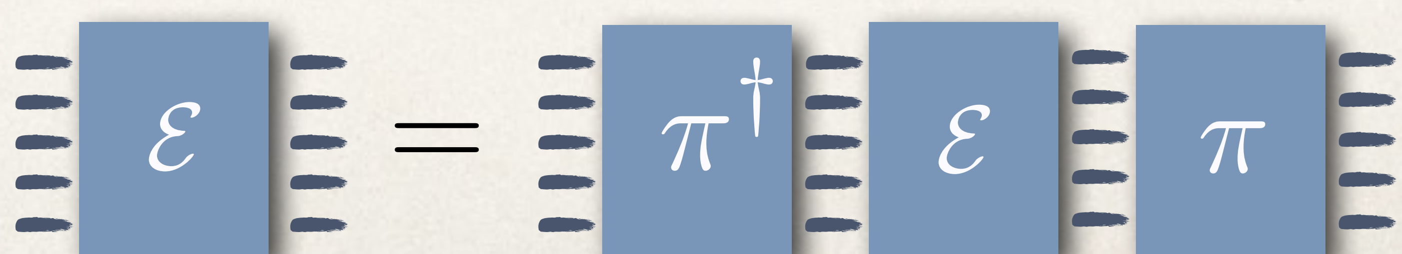
- Permutation covariant:



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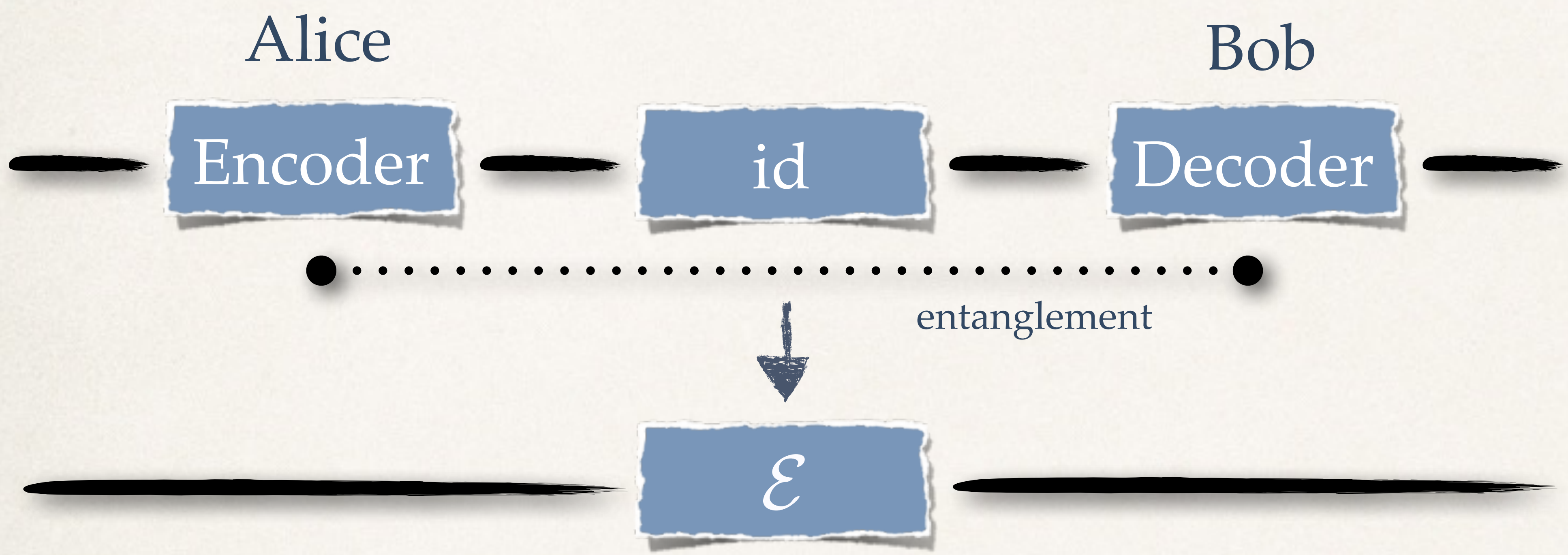
Conclusions



id: perfect classical channel
$\mathcal{E}$: quantum channel

Any quantum channel can be simulated by an unlimited amount of shared entanglement and an amount of classical communication equal to the channel's entanglement assisted classical capacity.

- Stinespring Dilation: $\mathcal{E}^{\otimes n}_{A\to B}(\rho^n_A) = \mathrm{tr}_{A'}(U^n_{A\to BA'}\rho^n_A U^n_{A\to BA'}) =: \mathrm{tr}_{A'}(\sigma^n_{BA'})$
- Local simulation of $U^n_{A\to BA'}$ and (optimal) one-shot State Splitting of $\sigma^n_{BA'}$ gives ε-approximation $\mathcal{F}^{n,\epsilon}_{A\to B}$ of $\mathcal{E}^{\otimes n}_{A\to B}$. Using Post-Selection Technique everything works!