Quantum adversaries via operator space theory

Mario Berta (IQIM Caltech), Omar Fawzi (ENS Lyon), Volkher Scholz (ETH Zurich) - partly based on arXiv:1409.3563

Caltech ENS DE LYON ETH zürich

Outline

- Motivation
- Randomness extraction against quantum adversaries
- Results - mathematical framework based on operator space theory
- Summary and outlook

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- Other examples: non-local boxes, quantum field theory, quantum gravity (?)
- **Goal**: understand similarities and differences

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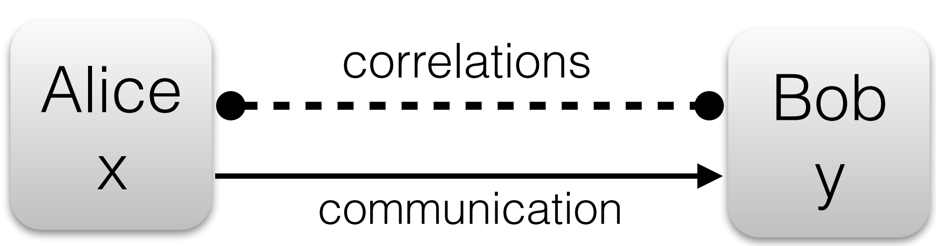

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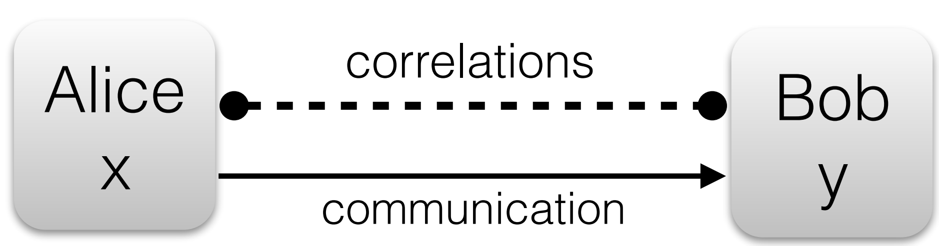

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- **Bell inequalities** (multi prover games): for a given game, what is the optimal winning probability (averaged over all possible questions)?

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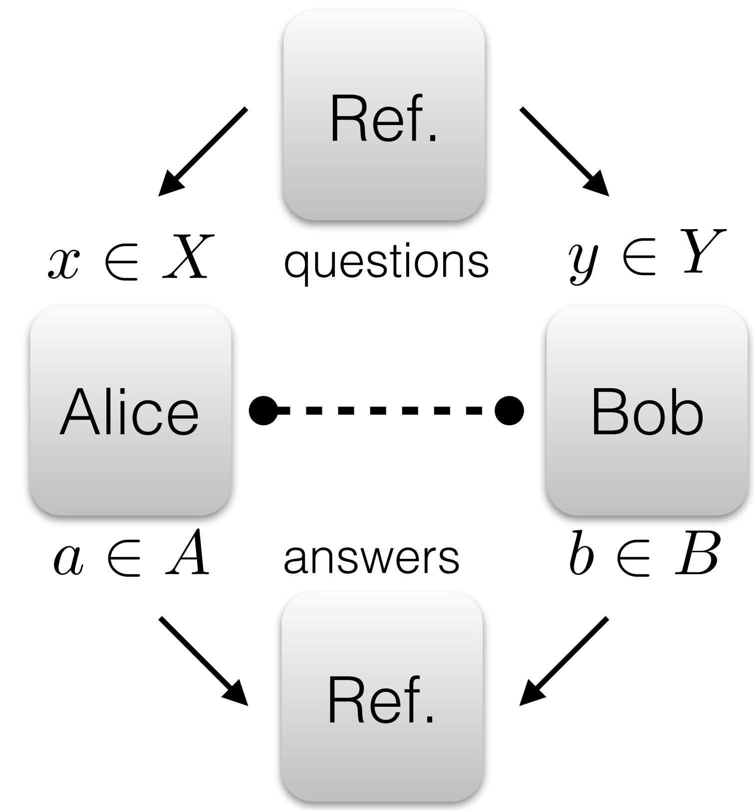

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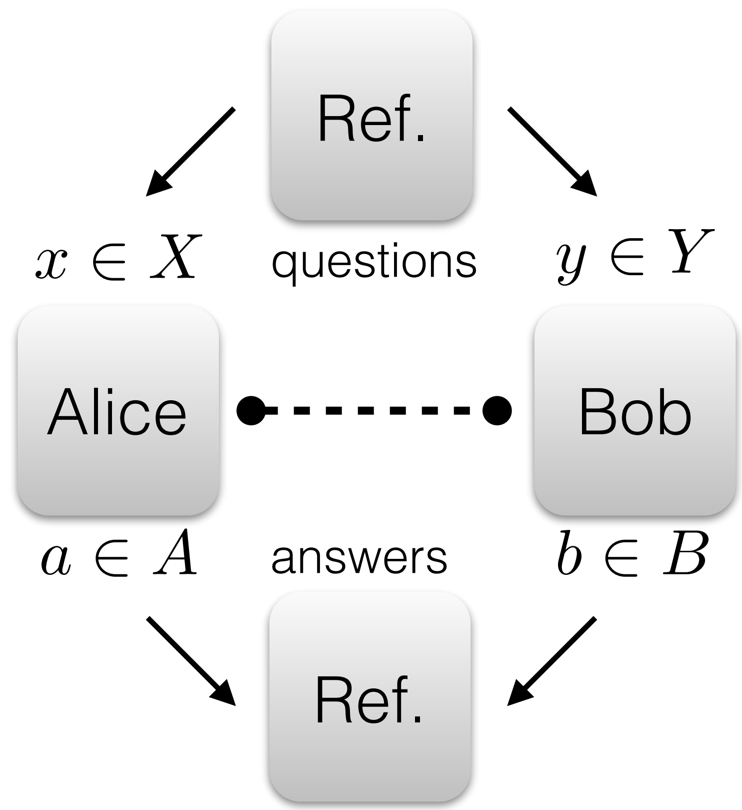

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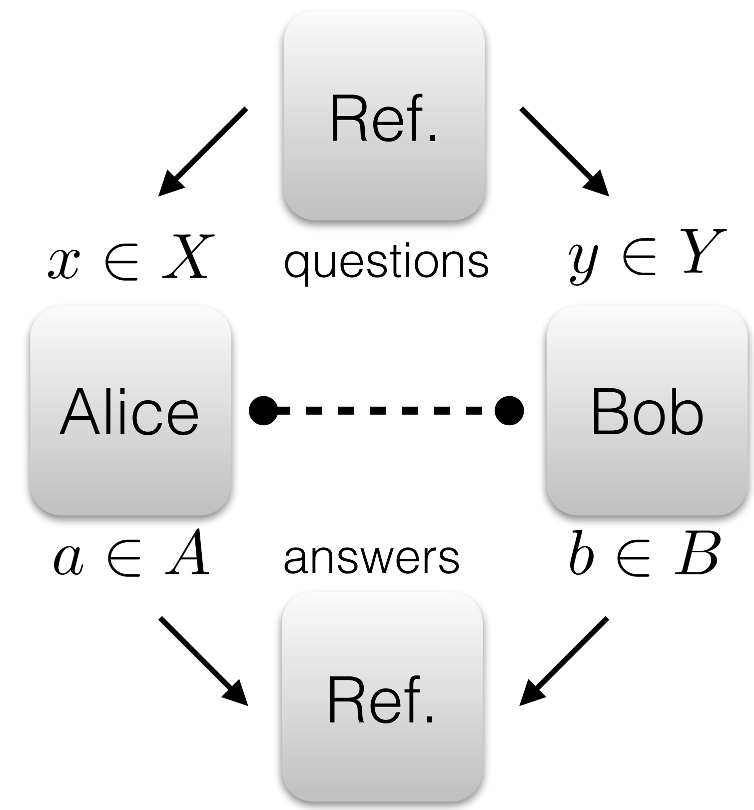

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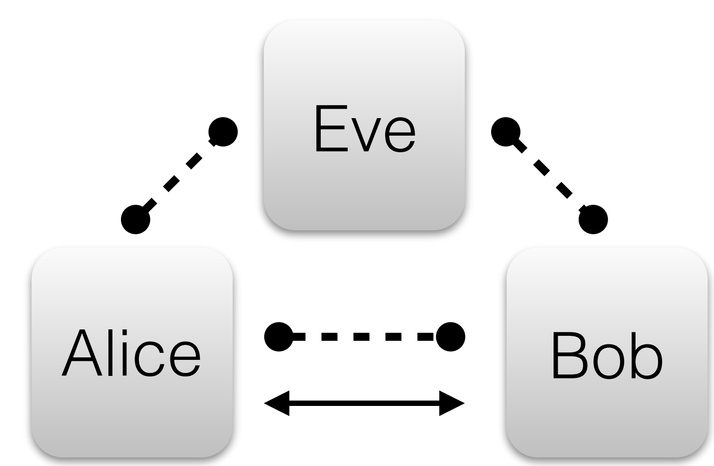

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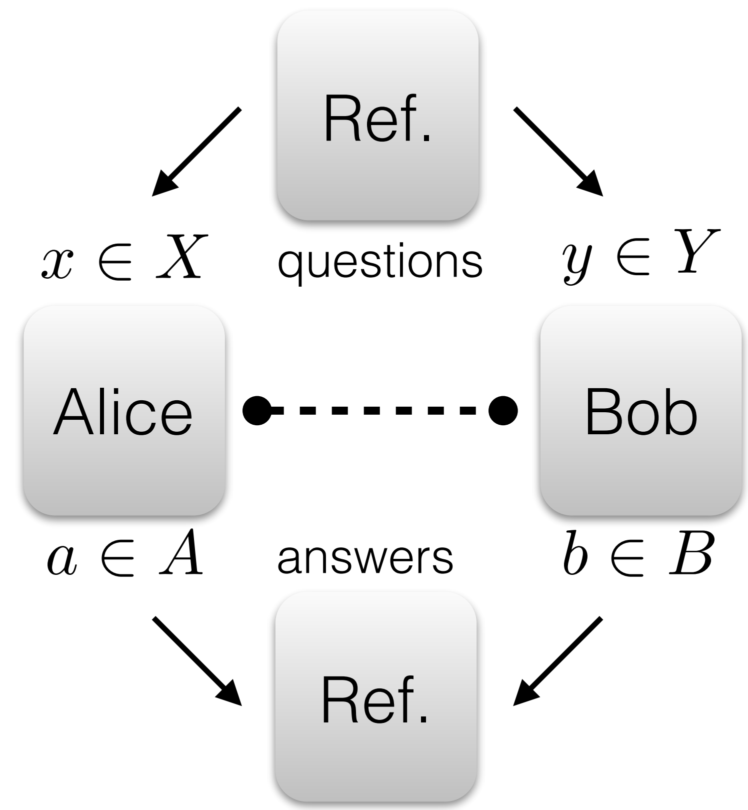

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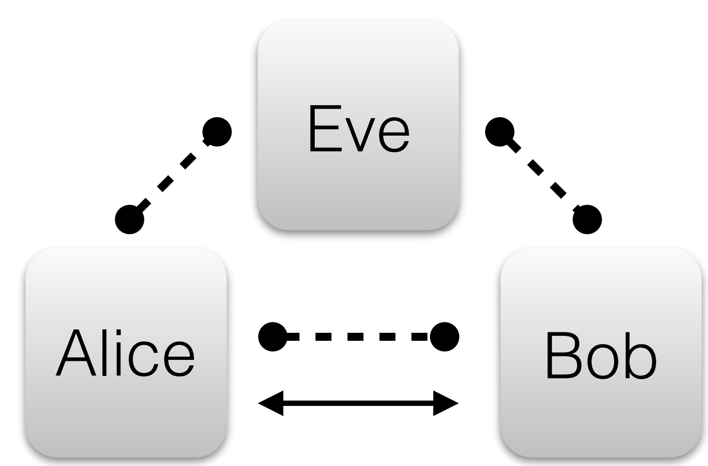

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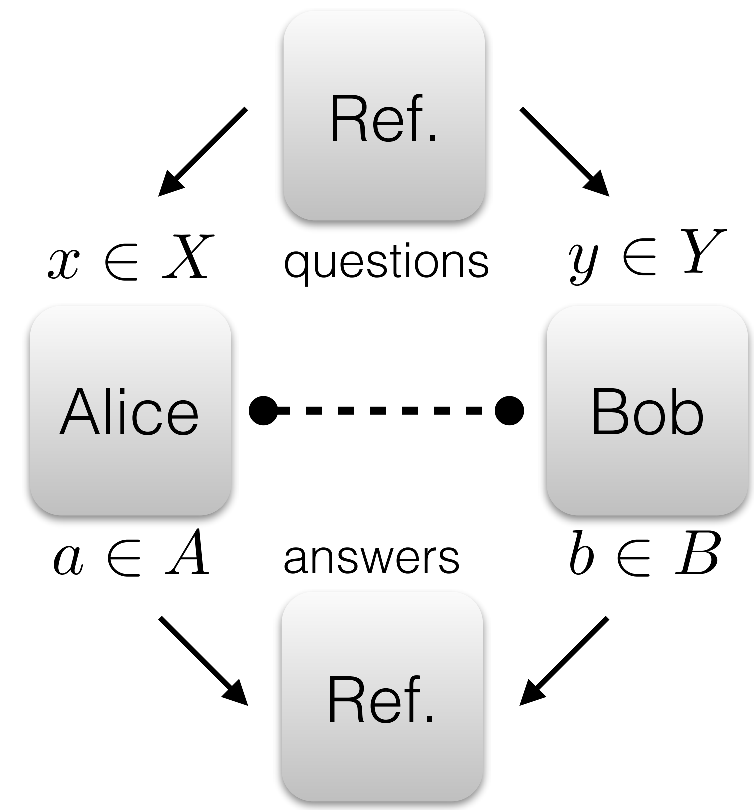

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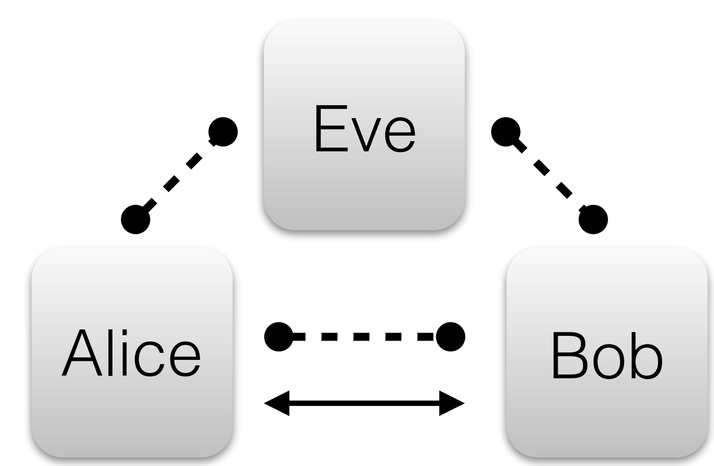
-> but also: quantum adversaries, post-quantum cryptography!



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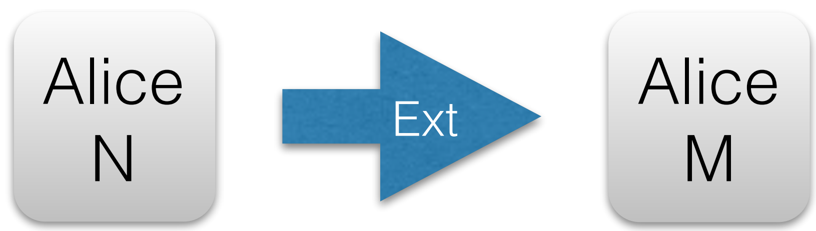
Randomness Extraction I

- **Goal**: transform only partly random classical source N into (almost perfectly) uniformly random source M (possibly over shorter alphabet)



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- **Condition**: contains some randomness as measured by

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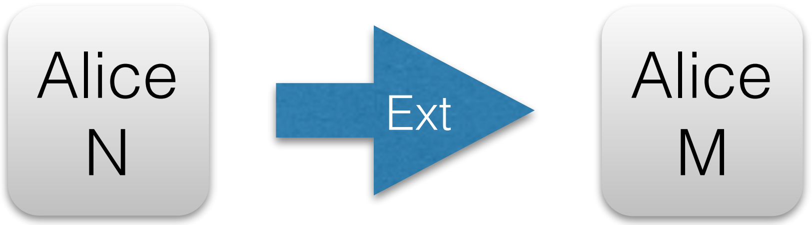

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- **Problem**: cannot be achieved in a deterministic way, if we require it to work for all sources satisfying the upper bound on the guessing probability

- **Solution**: can be achieved if the use of a catalyst is allowed, additional uniformly random source over alphabet $D = 2^d$ (called the seed)

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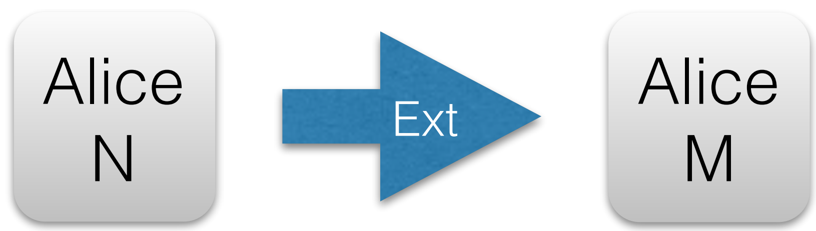

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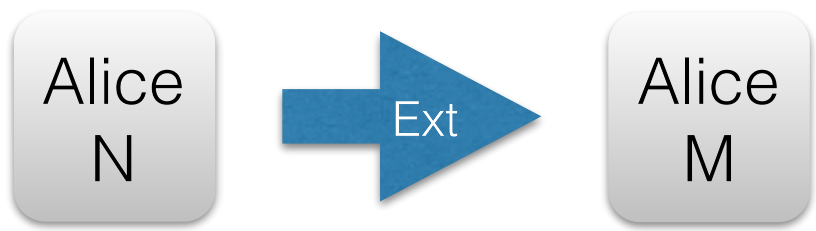

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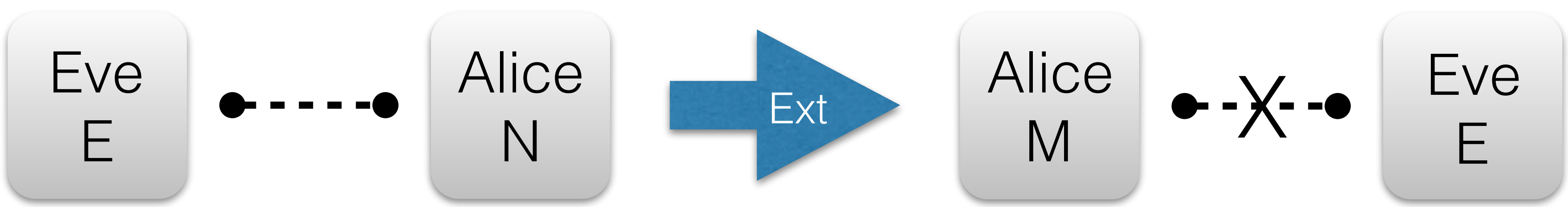

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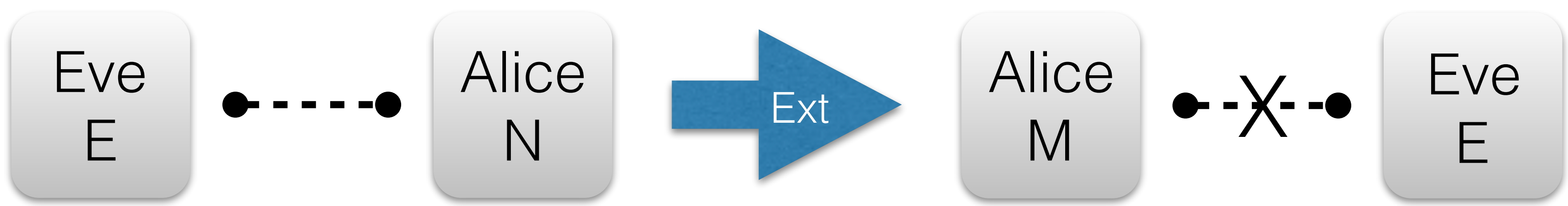

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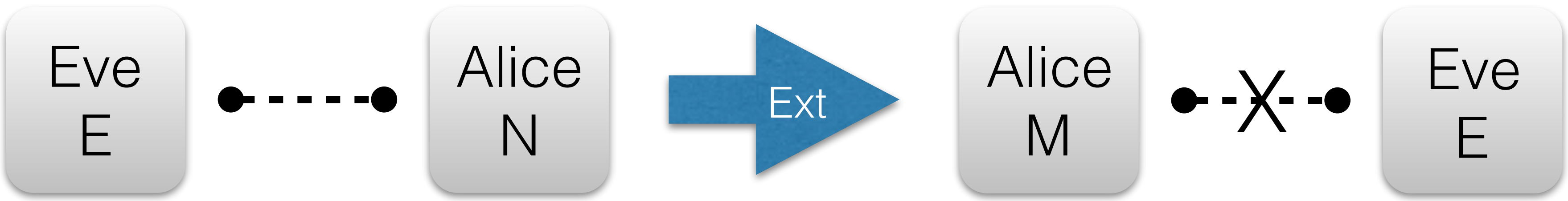

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- **Setup**: input is classical-quantum state with lower bound on the adversary's guessing probability of the secret N (given all her knowledge)

$$\rho_{NE} = \sum_{x \in N} |x\rangle\langle x|_N \otimes \rho_E^x \qquad p_{\text{guess}}(N|E)_\rho = \max_{\Lambda = \{\Lambda^x\}} \sum_{x \in N} \mathrm{tr}\left[\Lambda_E^x \rho_E^x\right] \leq 1/k$$

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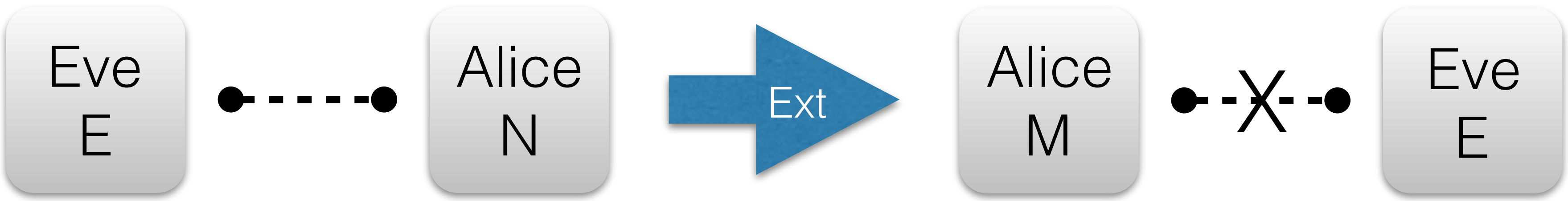

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- **Our work**: we developed mathematical framework to study this question based on ***operator space theory*** (cf. Bell inequalities)
- **Results**: derive all known result with unified proof strategy (using semi-definite program relaxations), plus give new bounds on the classical - quantum gap
- **Extra**: relate the question about the violation of Bell inequalities to the question about quantum-proof extractors

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Overview

$$C(\mathrm{Ext}, k) \quad \text{vs.} \quad Q(\mathrm{Ext}, k)$$

- Classical extractor property is expressed as ***norm*** of a linear mapping between ***normed linear spaces***

- These normed spaces can be ***quantised***, giving rise to ***operator spaces***

- The property ***quantum-proof*** extractor can be formulated in terms of a ***completely bounded norm*** (norms between operator spaces)

Linear Normed Spaces

- Consider the ***norm***: $\| \cdot \|_\cap = \max\{\| \cdot \|_1, k\| \cdot \|_\infty\}$

—> input constraint captured for distributions with $\|P\|_\cap \leq 1$

(remember: $C(\mathrm{Ext}, k) = \max_{p_{\mathrm{guess}}(N)_P \leq 1/k} \frac{1}{D} \sum_{i \in D} \|\mathrm{Ext}(i, P) - U_M\|_1 \leq \epsilon$)

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- Extractor characterised by ***linear mapping*** $\Delta[\text{Ext}] : \mathbb{R}^N \to \mathbb{R}^{DM}$:

$$\Delta[\text{Ext}](e_x) = \frac{1}{D} \sum_{\substack{i \in D \\ y \in M}} \left(\delta_{\text{Ext}(i,x)=y} - \frac{1}{M} \right) e_i \otimes e_y$$

with ***bounded norm*** constraint

$$\boxed{C(\text{Ext}, k) = \|\Delta[\text{Ext}]\|_{\cap \to 1} = \max\{\|\Delta[\text{Ext}](z)\|_1 : \|x\|_\cap \leq 1\| \leq \epsilon}$$

Operator Spaces

- Linear normed space W together with a ***sequence of norms*** on $W \otimes M_q,\ q \in \mathbb{N}$ satisfying some consistency conditions

classical

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- Analyse bounded vs. completely bounded norm: in general, but also for specific extractor constructions!

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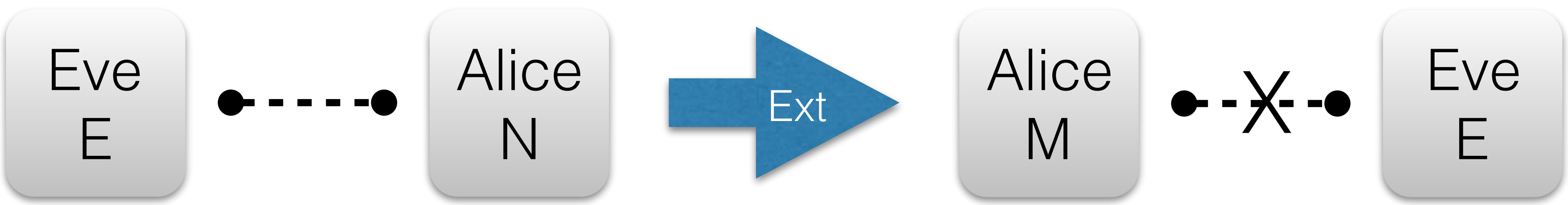

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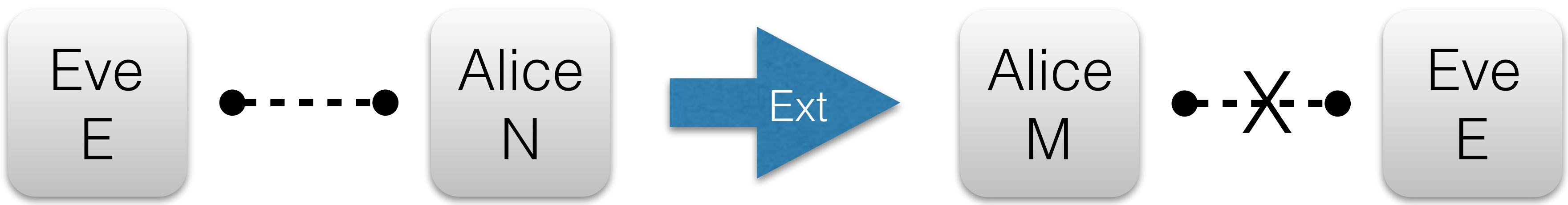

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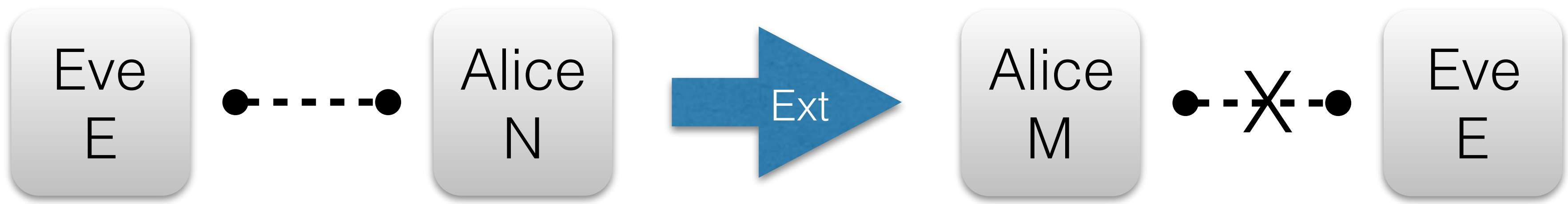

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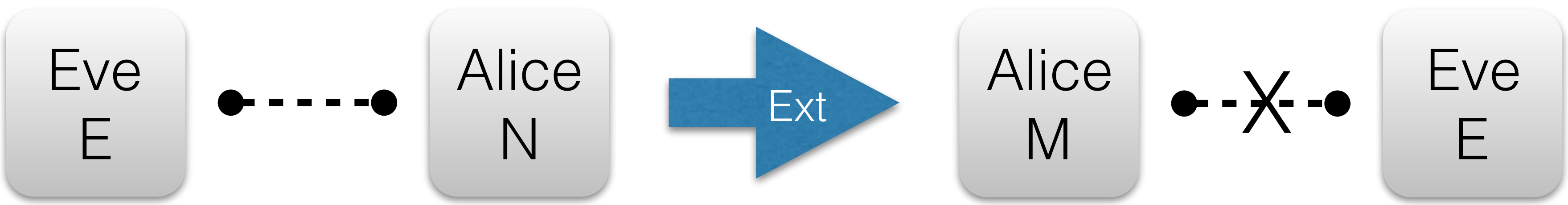

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Main question remains largely open!

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