Classical and Quantum Channel Simulations

Mario Berta (based on joint work with Fernando Brandão, Matthias Christandl, Renato Renner, Joseph Renes, Stephanie Wehner, Mark Wilde)

Outline

- Classical Shannon Theory

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  - Classical Shannon Theory
- Classical Channel Simulations:
  - Classical Reverse Shannon Theorem

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  - Entanglement Cost of Quantum Channels

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    - Post-Selection Technique for (Quantum) Channels
    - Randomness Extractors with (Quantum) Side Information

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- At what rate can a channel Λ simulate the identity channel?

Transmitter Alice

Receiver Bob

Λ: channel

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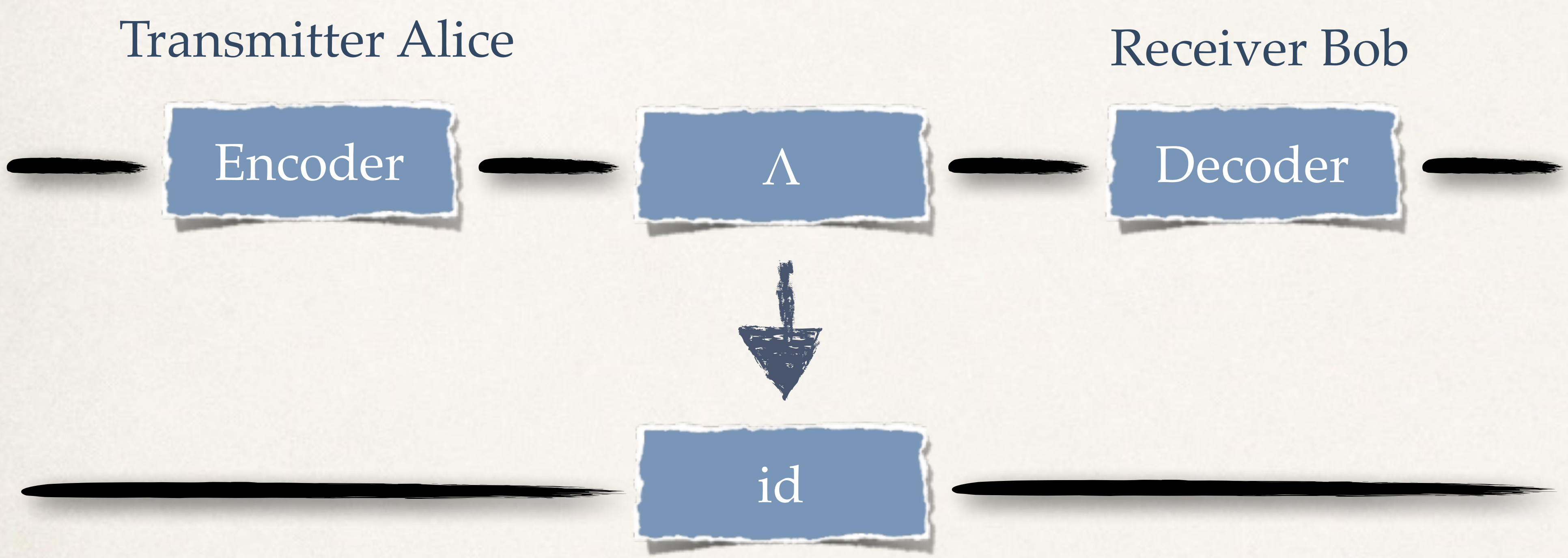

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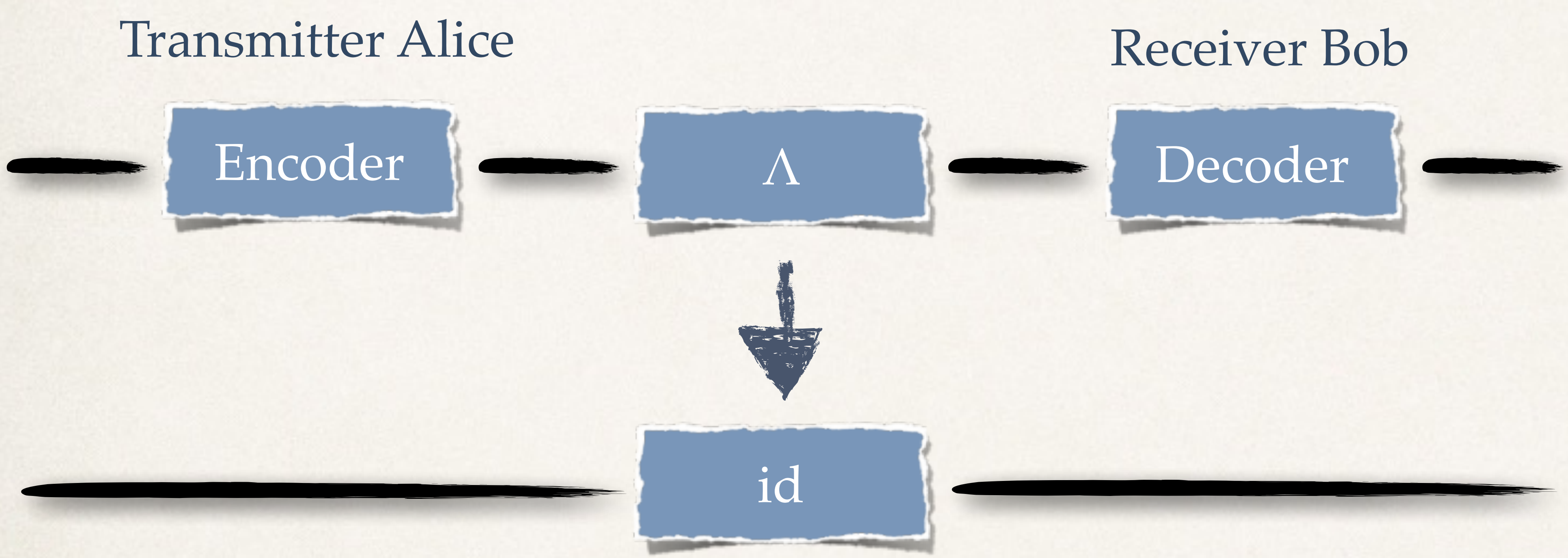

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- Shannon's noisy channel coding theorem, channel capacity [1]:

$$C(\Lambda) = \max_X I(X:\Lambda(X)) \quad I(X:Y) = H(X) + H(Y) - H(XY) \quad H(X) = -\sum_x p_x \log p_x$$

[1] Shannon, Bst J 27:379,623, 1948

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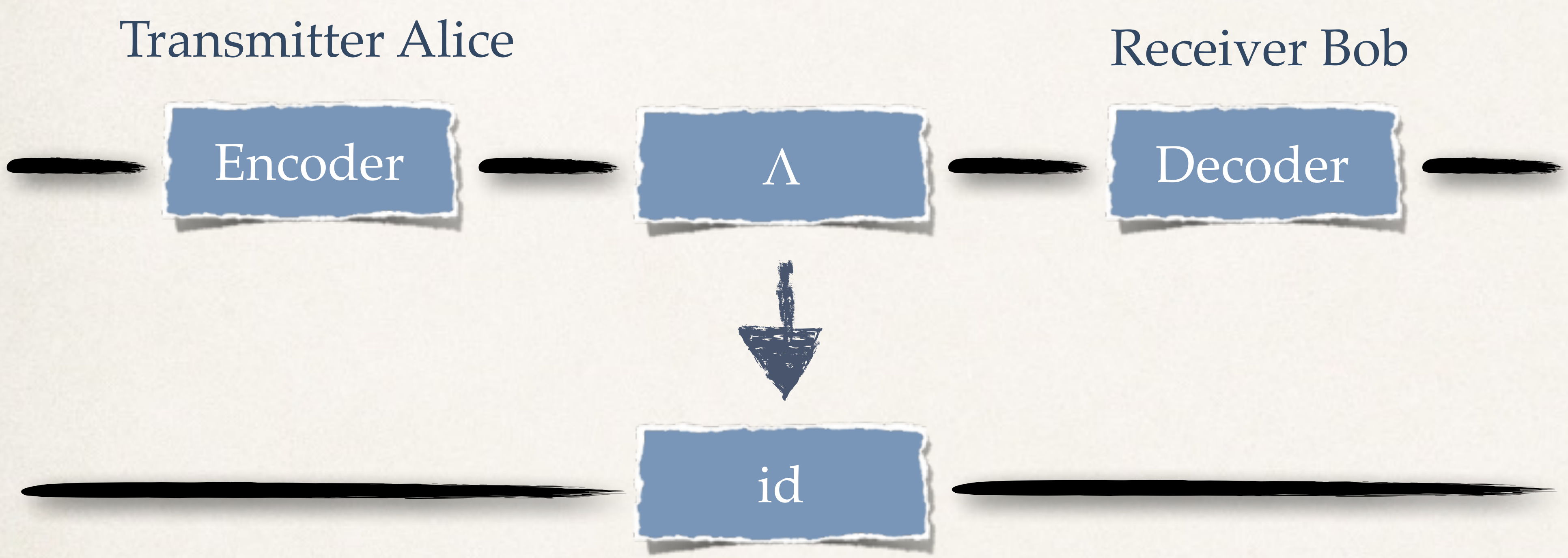

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- Neither back communication nor shared randomness help.

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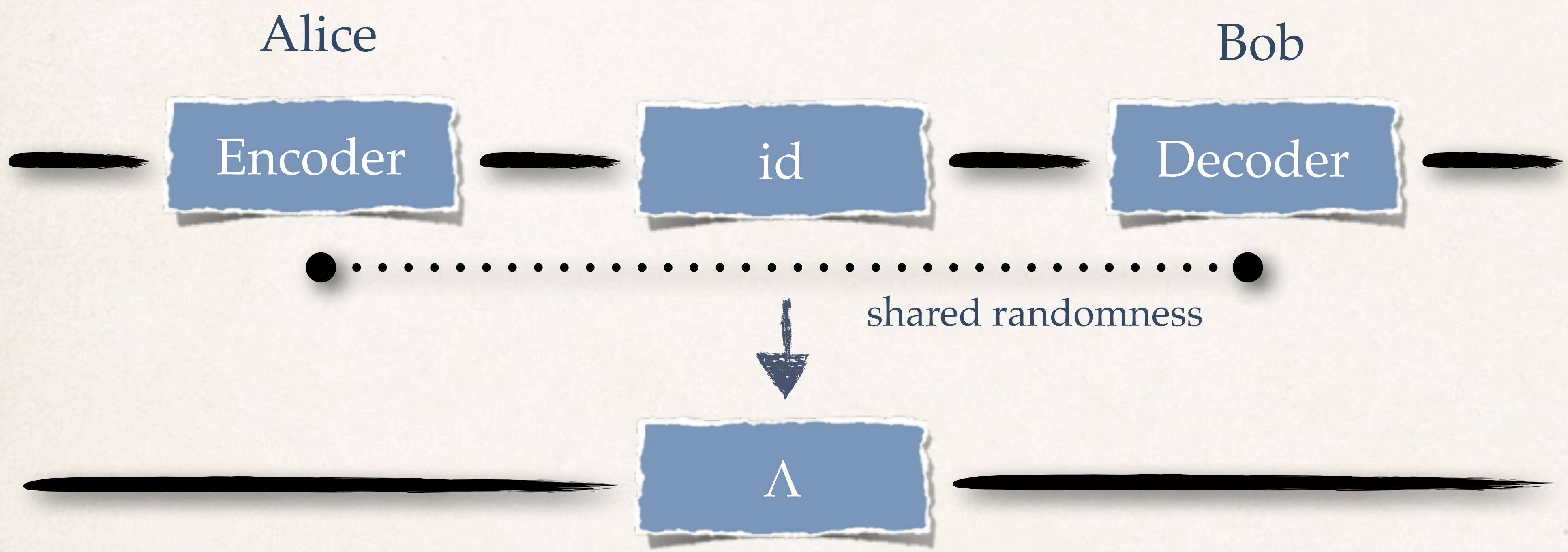
- At what rate can the identity channel simulate a channel Λ (using shared randomness)?



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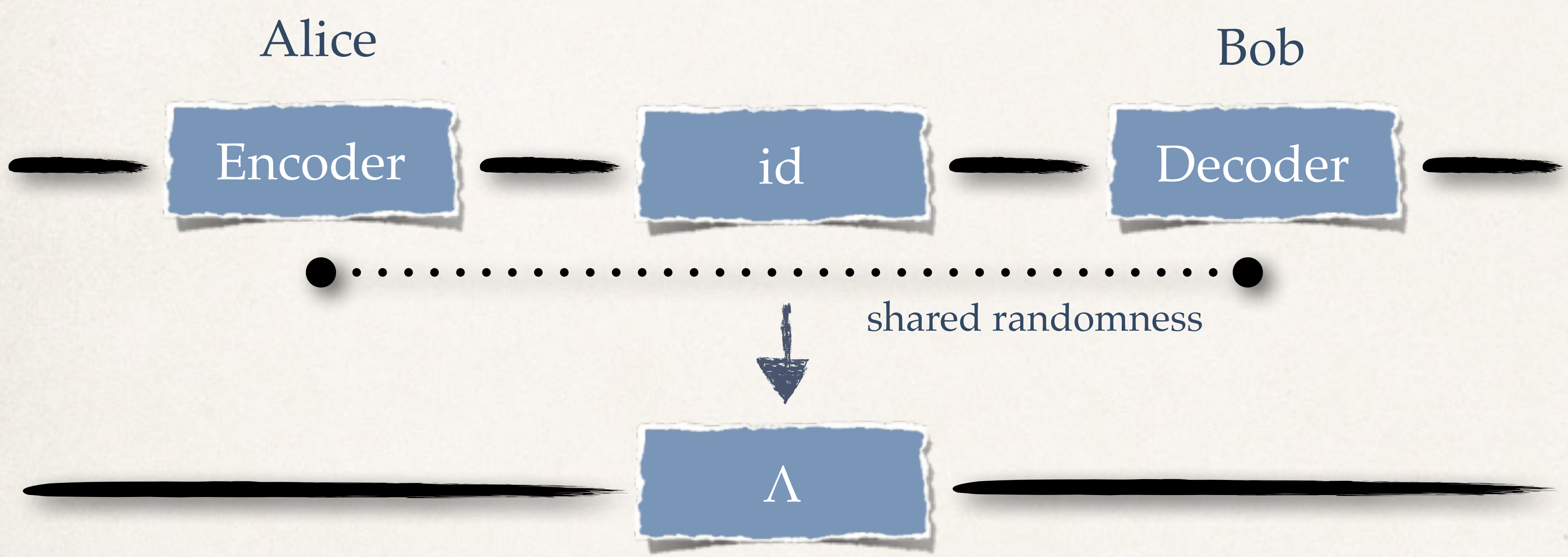

id: identity channel
Λ: channel

- Classical reverse Shannon theorem, channel simulation is possible if and only if [2,3]:

$$c \geq \max_X I(X : \Lambda(X)) \quad c + r \geq \max_X H(\Lambda(X))$$

$$C_{CRST}(\Lambda) = \max_X I(X : \Lambda(X)) = C(\Lambda)$$

[2] Bennett et al., IEEE TIF 48(10):2637, 2002
[3] Bennett et al., arXiv:0912.5537v2

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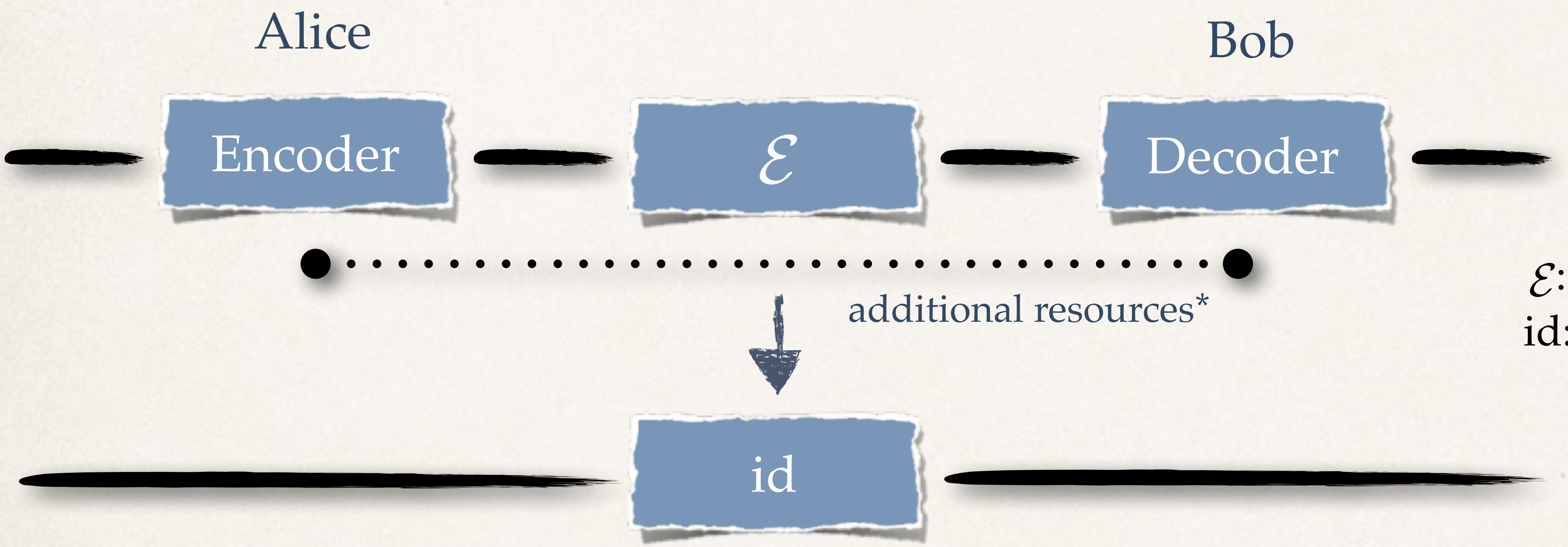

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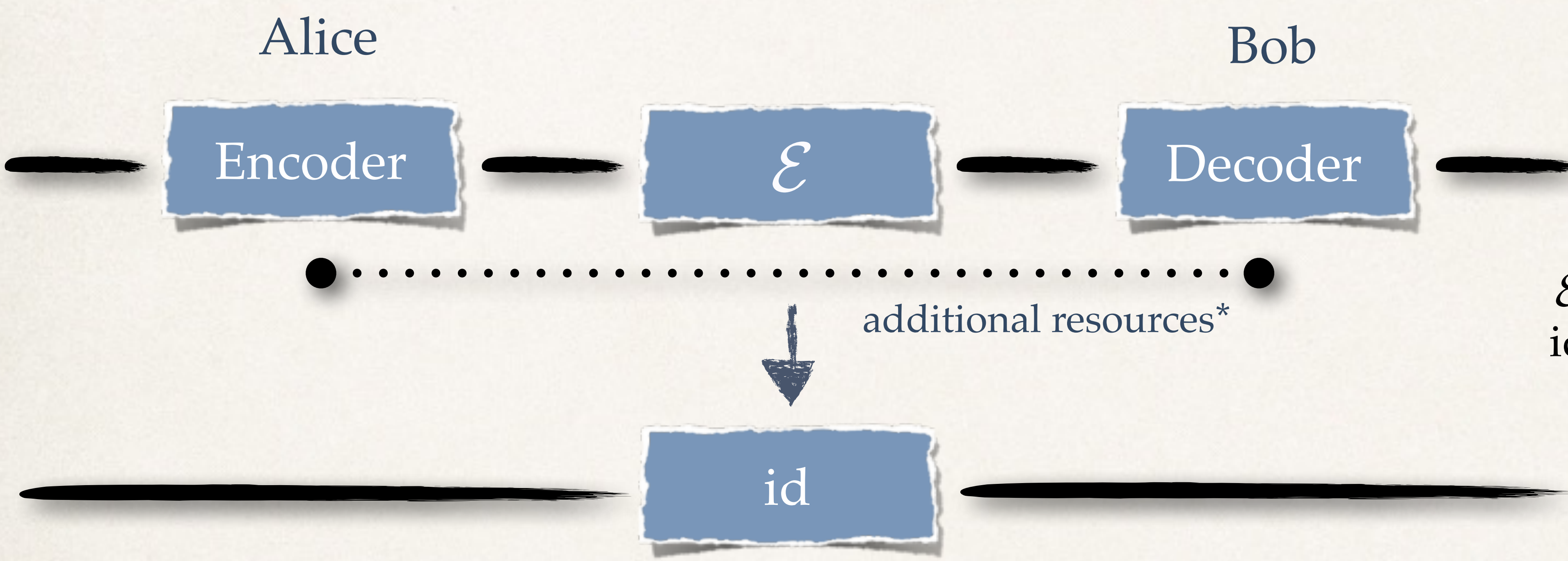

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- Quantum Channel Capacities [...]: $Q, Q_E, Q_\rightarrow, Q_\leftarrow, Q_\leftrightarrow$

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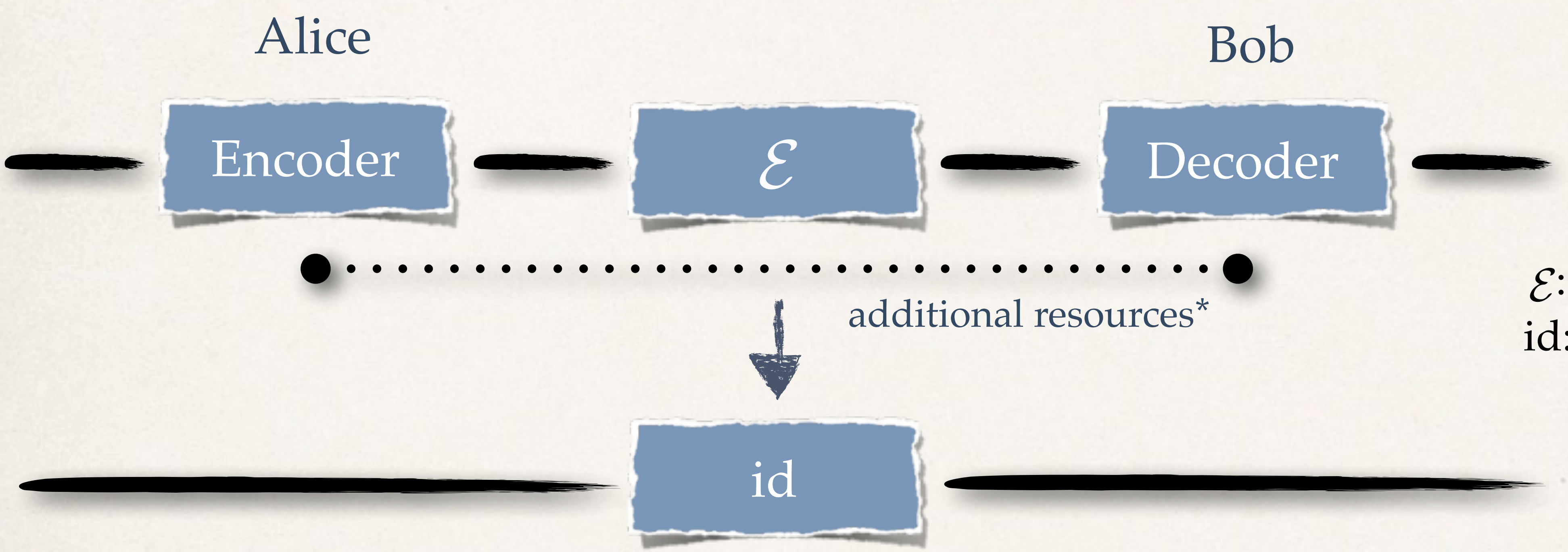

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- Entanglement-assisted quantum capacity [2]:

$$I(A:B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho$$

$$H(A)_\rho = -\mathrm{tr}[\rho_A \log \rho_A]$$

$$Q_E(\mathcal{E}) = \frac{1}{2} \cdot \max_\rho I(B:R)_{(\mathcal{E}\otimes \mathrm{id})(\Phi_\rho)}$$

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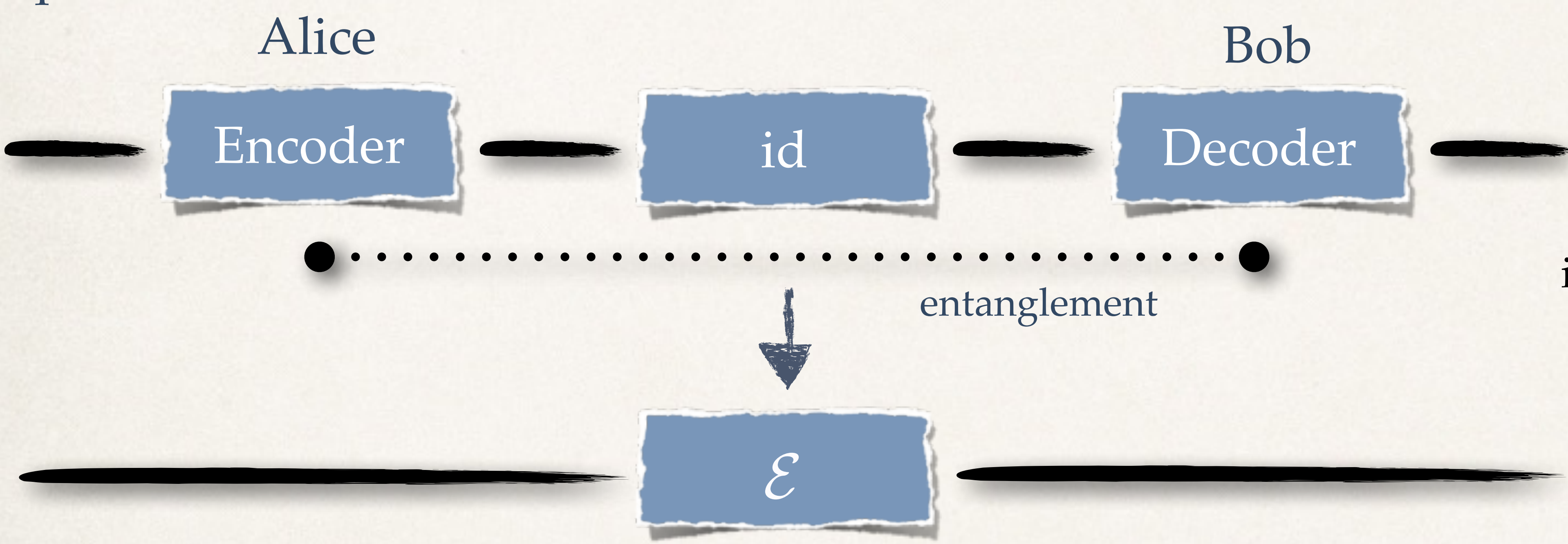
- Using entanglement, at what rate can the quantum identity channel simulate a quantum channel?



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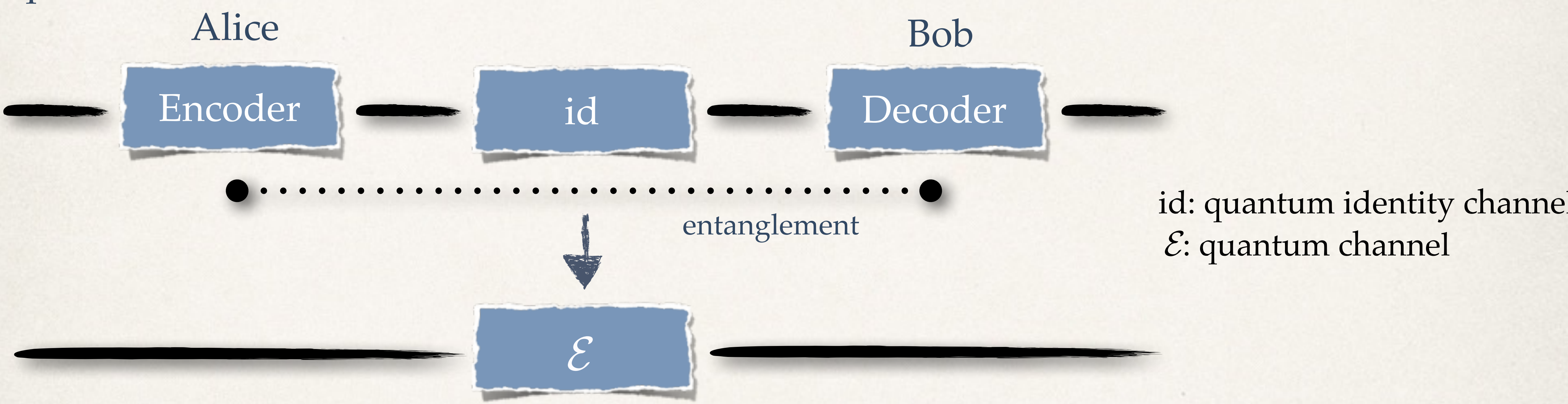

- Quantum reverse Shannon theorem, channel simulation is possible for [3,4]:

$$q = \frac{1}{2} \cdot \max_{\rho} I(B:R)_{(\mathcal{E}\otimes\mathrm{id})(\Phi_\rho)} \qquad e = \infty \qquad \text{(embezzling states)}$$

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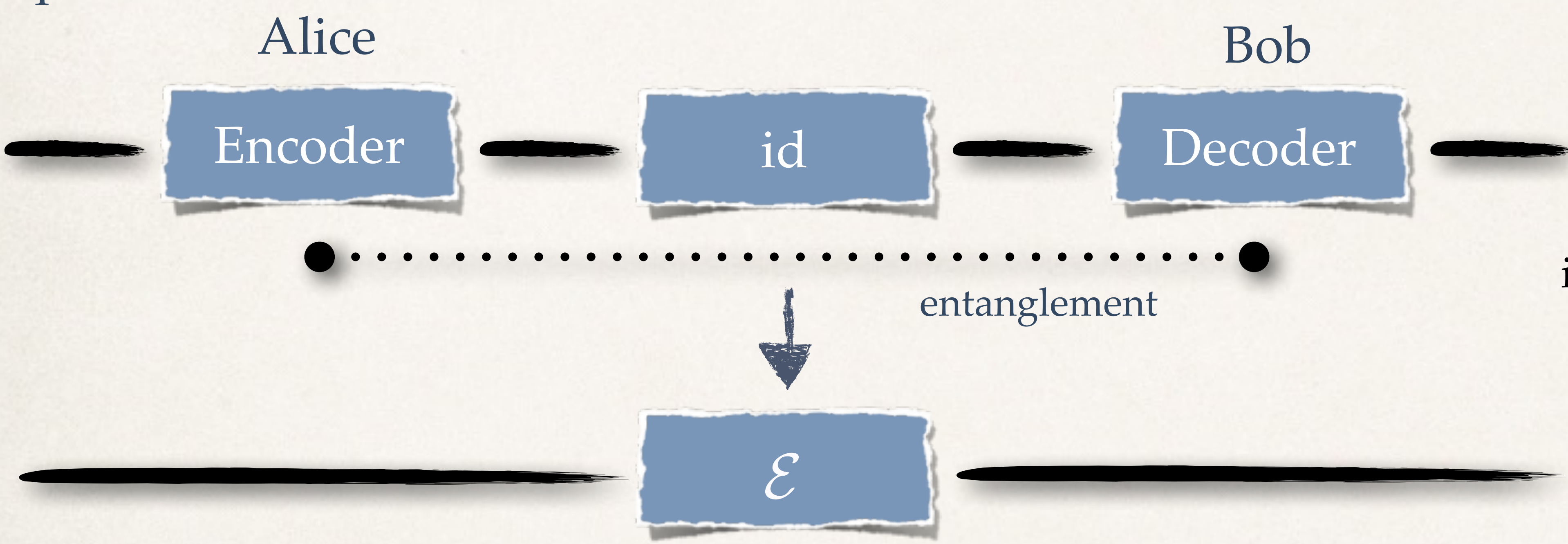

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$$Q_{QRST}(\mathcal{E}) = \frac{1}{2} \cdot \max_{\rho} I(B:R)_{(\mathcal{E}\otimes \mathrm{id})(\Phi_\rho)} = Q_E(\mathcal{E})$$

- Communication optimal, for more communication other tradeoffs are possible [3].

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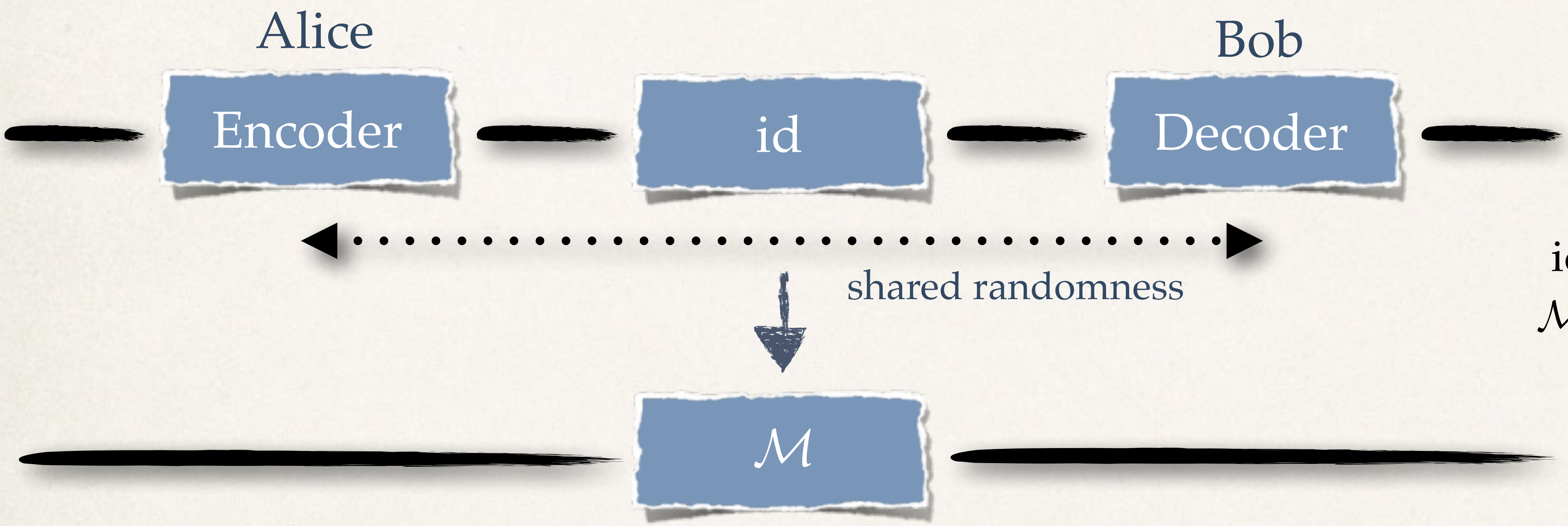
Information Gain of Quantum Measurements

- At what rate can the classical identity channel simulate a quantum measurement (using shared randomness)?



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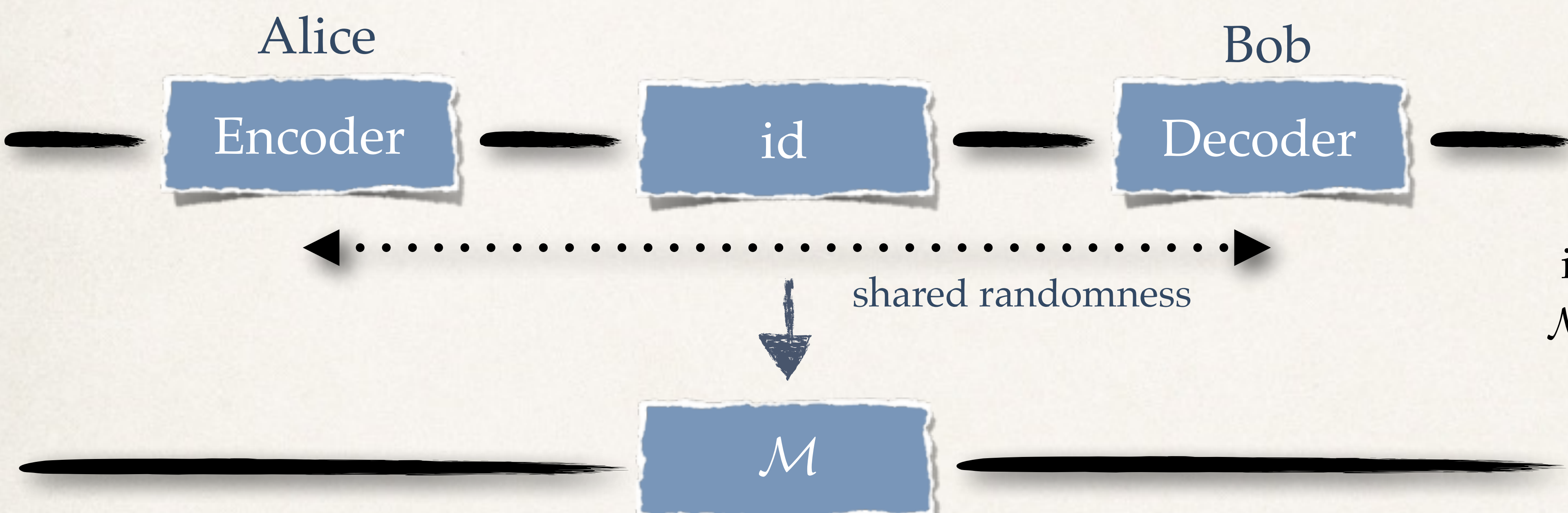

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- Measurement simulation is possible if and only if (universal measurement compression) [5]:

$$c \geq \max_{\rho} I(X_B : R)_{(\mathcal{M}\otimes \mathrm{id})(\Phi_\rho)} \qquad c + r \geq \max_{\rho} H(X_B)_{\mathcal{M}(\rho)}$$

[5] Berta et al., arXiv:1301.1594V1

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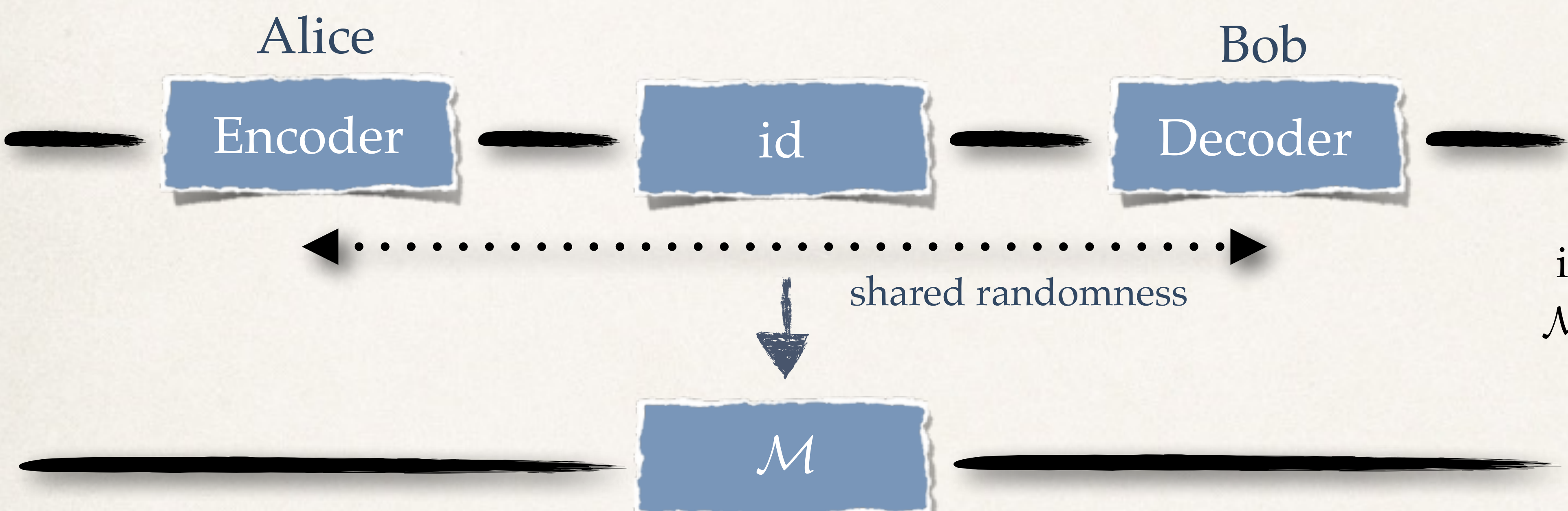

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$$C_{IG}(\mathcal{M}) = \max_{\rho} I(X_B : R)_{(\mathcal{M}\otimes\mathrm{id})(\Phi_\rho)} \qquad (= C_E(\mathcal{M}))$$

- Following Winter [6]: $C_{IG}(\mathcal{M})$ is the information gained by the measurement!

[5] Berta et al., arXiv:1301.1594V1

[6] Winter, CMP 244(1):157, 2004

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Entanglement Cost of Quantum Channels

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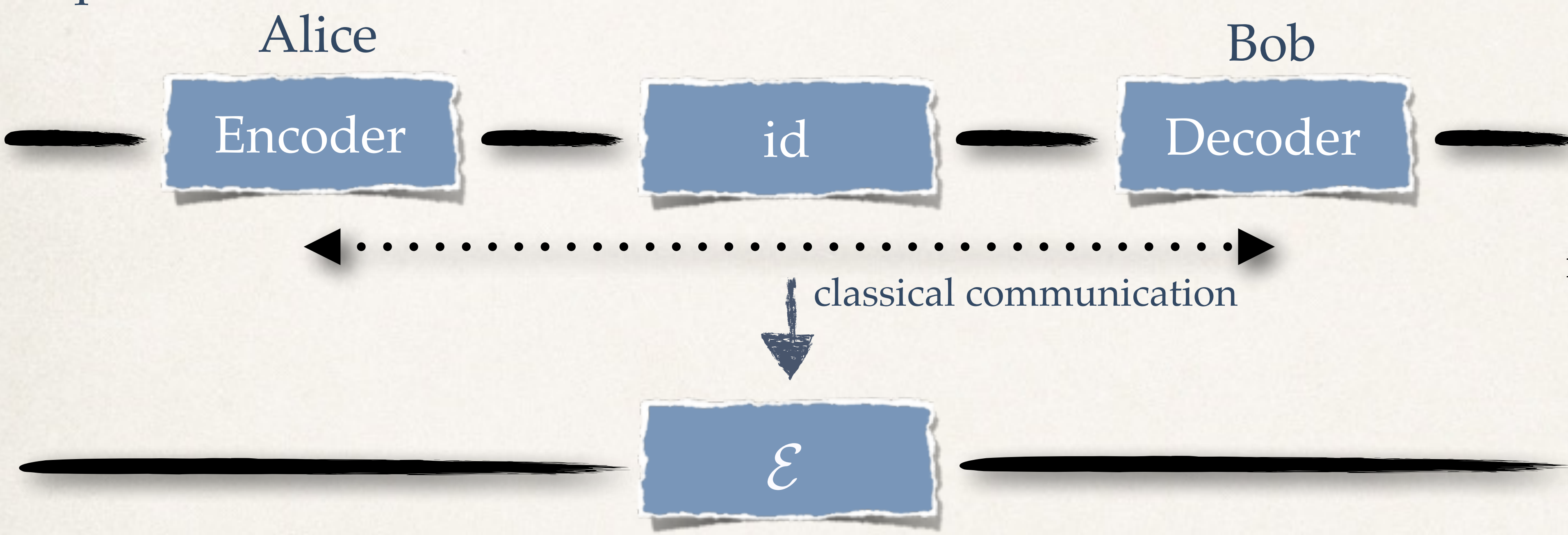

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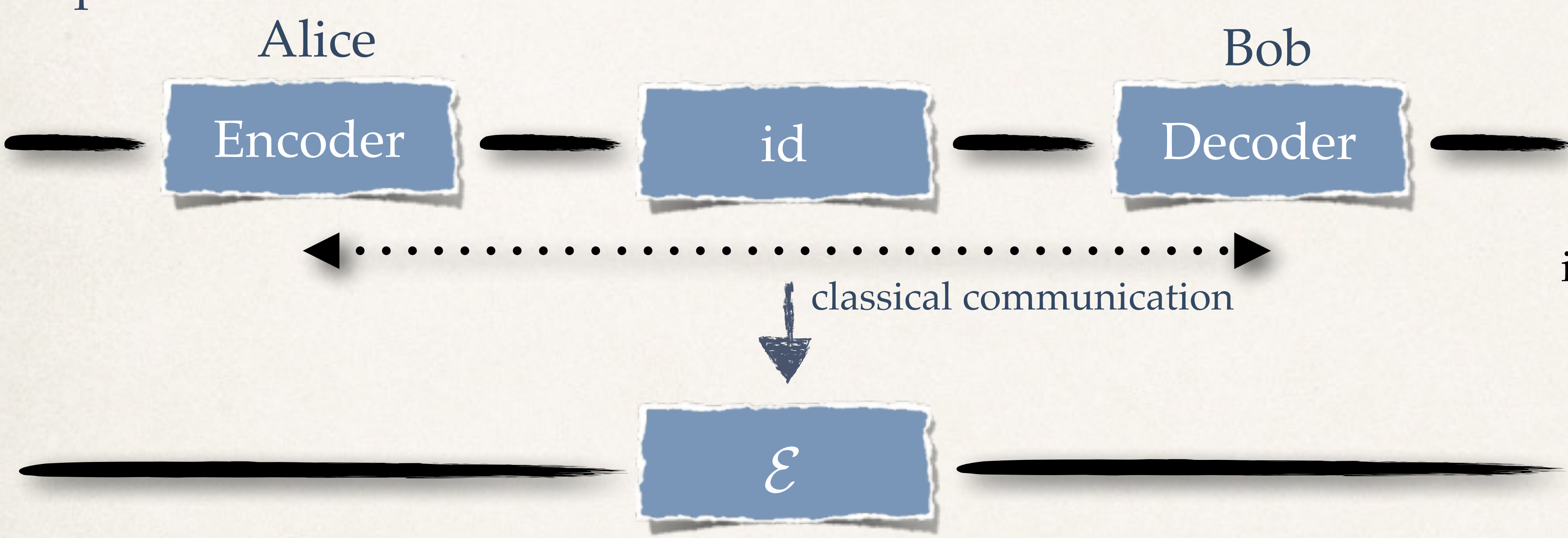

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- Quantum communication equivalent to ebits, channel simulation possible for [7]:

$$\boxed{q = e = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^n} E_F((\mathcal{E}^{\otimes n} \otimes \mathcal{I})(\Phi_{\rho^n}))} \quad E_F(\rho_{AB}) = \inf_{\{p_i,\rho^i\}} \sum_i p_i H(A)_{\rho^i} \quad \rho_{AB} = \sum_i p_i \rho^i_{AB}$$

$c = \infty$ (unknown)

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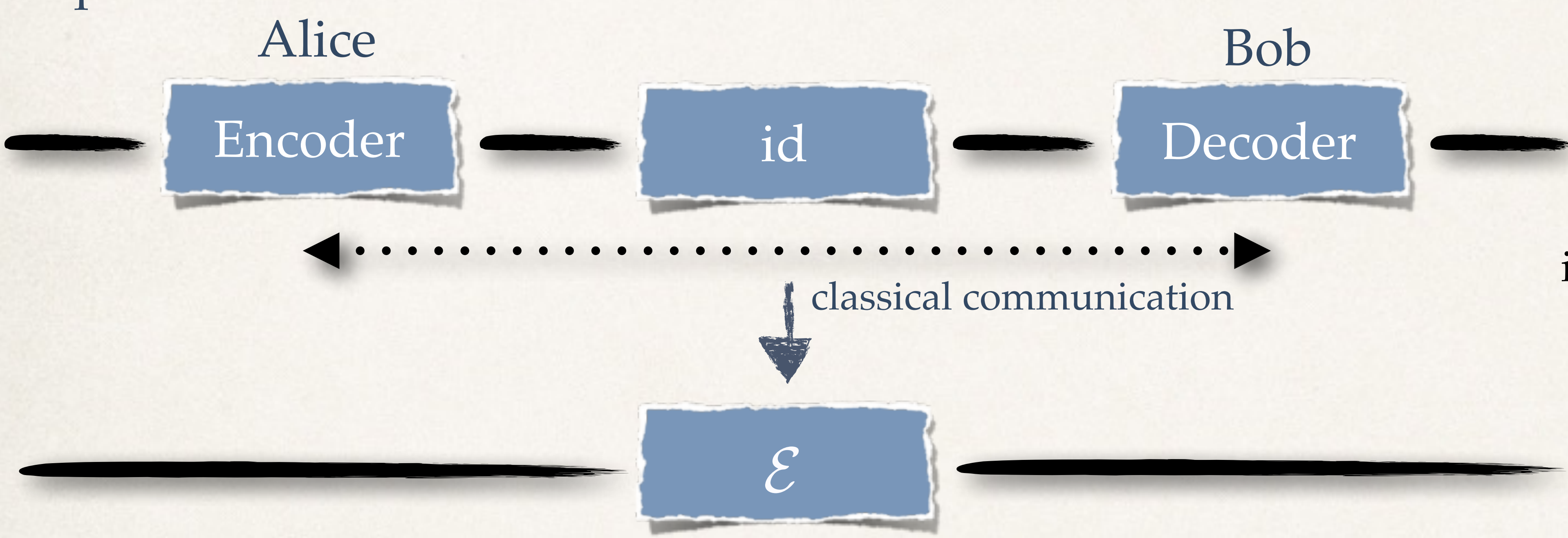

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$c = \infty$ (unknown)

$$E_C(\mathcal{E}) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^n} E_F((\mathcal{E}^{\otimes n} \otimes \mathcal{I})(\Phi_{\rho^n})) \quad (\geq Q_{\leftrightarrow}(\mathcal{E}))$$

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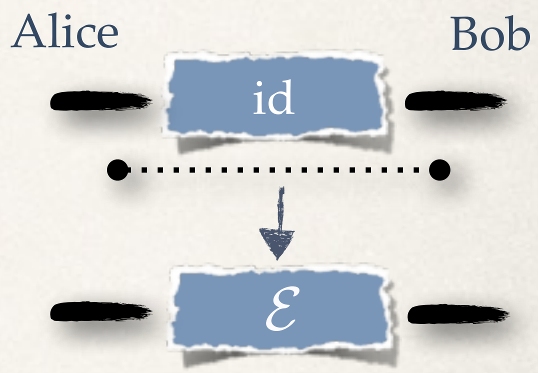
Proof Idea: Post-Selection Technique for Channels

- $\mathcal{E}_{A\to B}^{\otimes n}$: to simulate $\mathcal{F}_{A\to B}^{n,\varepsilon}$: channel simulation with cost $x^{(1)}(\mathcal{F}_{A\to B}^{n,\varepsilon})$



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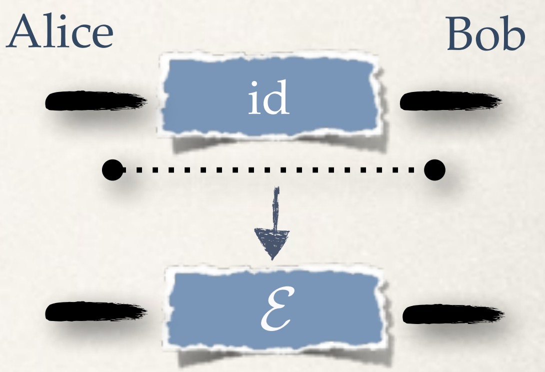
- Channel Simulation has to work for all (entangled) inputs!



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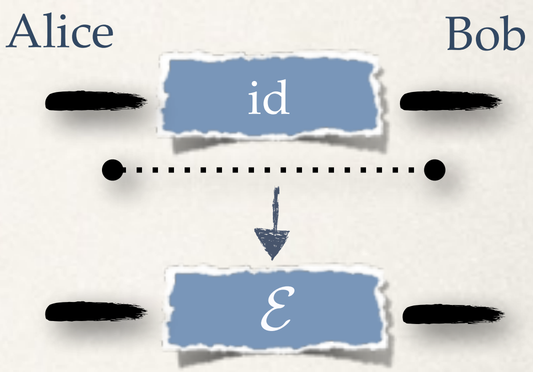

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- The diamond norm of a quantum operation is defined as

$$\|\mathcal{E}\|_{\diamondsuit} = \sup_{k\in\mathbb{N}} \sup_{\|\sigma\|_1\leq 1} \|(\mathcal{E}\otimes \mathrm{id}_k)(\sigma)\|_1 \quad \|\sigma\|_1 = \mathrm{tr}(\sqrt{\sigma^\dagger\sigma})$$

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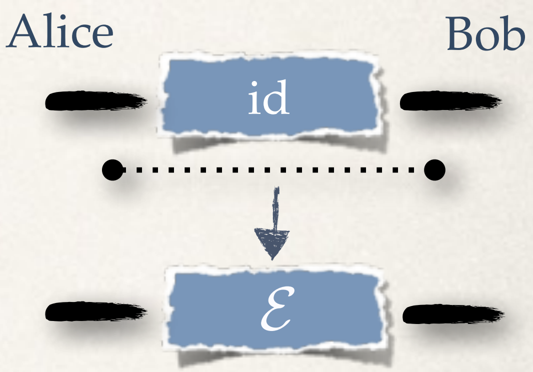
- We need: $\lim_{\epsilon\to 0}\lim_{n\to\infty} \|\mathcal{E}_{A\to B}^{\otimes n} - \mathcal{F}_{A\to B}^{n,\epsilon}\|_\diamond = 0 \quad \lim_{\epsilon\to 0}\lim_{n\to\infty} \frac{1}{n} x^{(1)}(\mathcal{F}_{A\to B}^{n,\varepsilon}) = X(\mathcal{E})$



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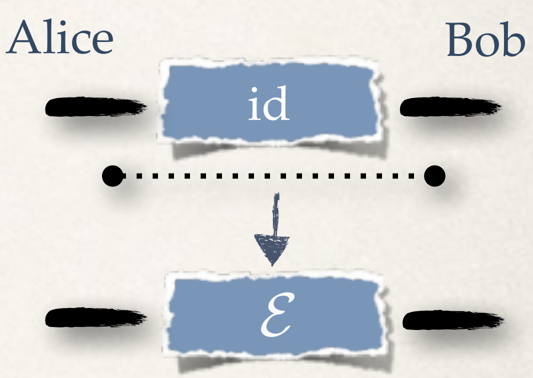

- $\mathcal{E}^{\otimes n}_{A\to B}$: to simulate $\mathcal{F}^{n,\varepsilon}_{A\to B}$: channel simulation with cost $x^{(1)}(\mathcal{F}^{n,\varepsilon}_{A\to B})$
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- Post-selection technique for quantum channels [8]:

$$\|\mathcal{E}^{\otimes n} - \mathcal{F}^{n,\varepsilon}\|_\diamond \leq \mathrm{poly}(n)\cdot \|((\mathcal{E}^{\otimes n} - \mathcal{F}^{n,\varepsilon})\otimes \mathrm{id})(\zeta^n)\|_1$$

ζ^n is the purification of a special de Finetti state (a state which consists of n identical and independent copies of a state on a single subsystem). No IID structure!

[8] Christandl et al., PRL 102(2):020504, 2009

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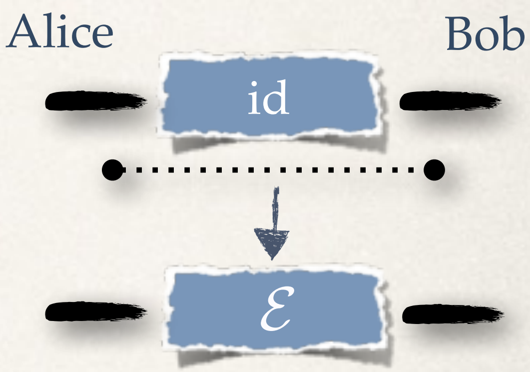

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- Basic idea: create $\sigma^n = \mathcal{E}^{\otimes n}(\zeta^n)$ locally at Alice's side, send it over to Bob's side. This defines the channel simulation $\mathcal{F}^{n,\varepsilon}$!

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Proof Idea: Randomness Extractors with Side Information

- State transfer from Alice to Bob: $\sigma^n = \mathcal{E}^{\otimes n}(\zeta^n)$

  Use random coding schemes and analyze how well they perform in the one-shot regime --> randomness extractors, also called decoupling (cf. Renner's and Dupuis' talk)!

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[9] Renner, PhD Thesis, ETHZ, 2005

[10] Tomamichel, PhD Thesis, ETHZ, 2011

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- One-shot information theory, smooth entropy formalism [9,10].
- Technically:
  - Classical reverse Shannon theorem: classical **randomness extractor** with classical side information (e.g. random permutation).
  - Information gain of quantum measurements: classical **randomness extractor** with quantum side information (e.g. random permutation).
  - Entanglement cost of quantum channels: quantum **randomness extractor** (e.g. random unitary).
  - Quantum reverse Shannon theorem: quantum **randomness extractor** with quantum side information (e.g. random unitary).

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  - Information Gain of Quantum Measurements
  - Entanglement Cost of Quantum Channels
- Proof Idea:
  - Post-Selection Technique for (Quantum) Channels
  - Randomness Extractors with (Quantum) Side Information
- Extensions and Applications

Extensions and Applications

- Results:

$C_{CRST}(\Lambda) = \max_X I(X : \Lambda(X))$ — classical reverse Shannon

$C_{IG}(\mathcal{M}) = \max_\rho I(X_B : R)_{(\mathcal{M}\otimes \mathrm{id})(\Phi_\rho)}$ — information gain of measurements

$Q_{QRST}(\mathcal{E}) = \frac{1}{2} \cdot \max_\rho I(B : R)_{(\mathcal{E}\otimes \mathrm{id})(\Phi_\rho)}$ — quantum reverse Shannon

$E_C(\mathcal{E}) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho^n} E_F((\mathcal{E}^{\otimes n} \otimes \mathcal{I})(\Phi_{\rho^n}))$ — entanglement cost

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- Feedback vs. non-feedback simulations (classical and quantum) [3,5].

[3] Bennett et al., arXiv:0912.5537v2

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- That's it...

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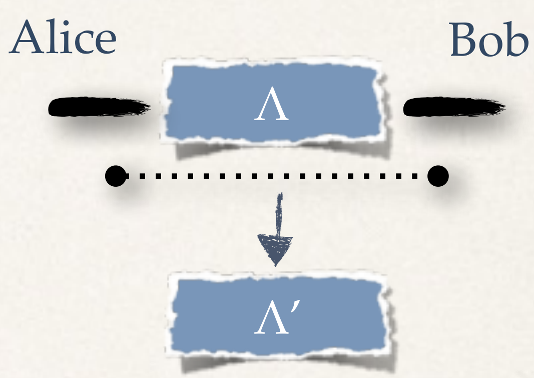
Example for Classification of Channels

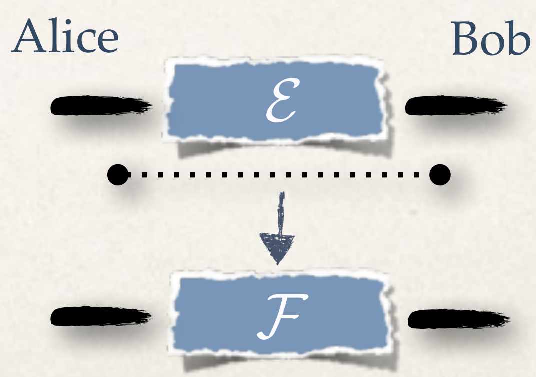
Capacity of a classical channel Λ to simulate another classical channel Λ' in the presence of free shared randomness is given by:

$$C_R(\Lambda, \Lambda') = \frac{C(\Lambda)}{C(\Lambda')}$$



Capacity of a quantum channel to simulate another quantum channel in the presence of free entanglement is given by:

$$C_E(\mathcal{E}, \mathcal{F}) = \frac{C_E(\mathcal{E})}{C_E(\mathcal{F})}$$



Example of Embezzling States

- Introduced by Van Dam and Hayden [11]
- **Definition:** A pure, bipartite state of the form

$$|\mu(k)\rangle_{AB} = \frac{1}{\sqrt{G(k)}} \sum_{j=1}^{k} \frac{1}{\sqrt{j}} |jj\rangle_{AB}$$

where $G(k) = \sum_{j=1}^{k} \frac{1}{j}$, is called *embezzling state* of index *k*.

- **Proposition:** Let $\epsilon > 0$ and let $|\varphi\rangle_{AB}$ be a pure bipartite state of Schmidt rank *m*. Then the transformation

$$|\mu(k)\rangle_{AB} \mapsto |\mu(k)\rangle_{AB} \otimes |\varphi\rangle_{AB}$$

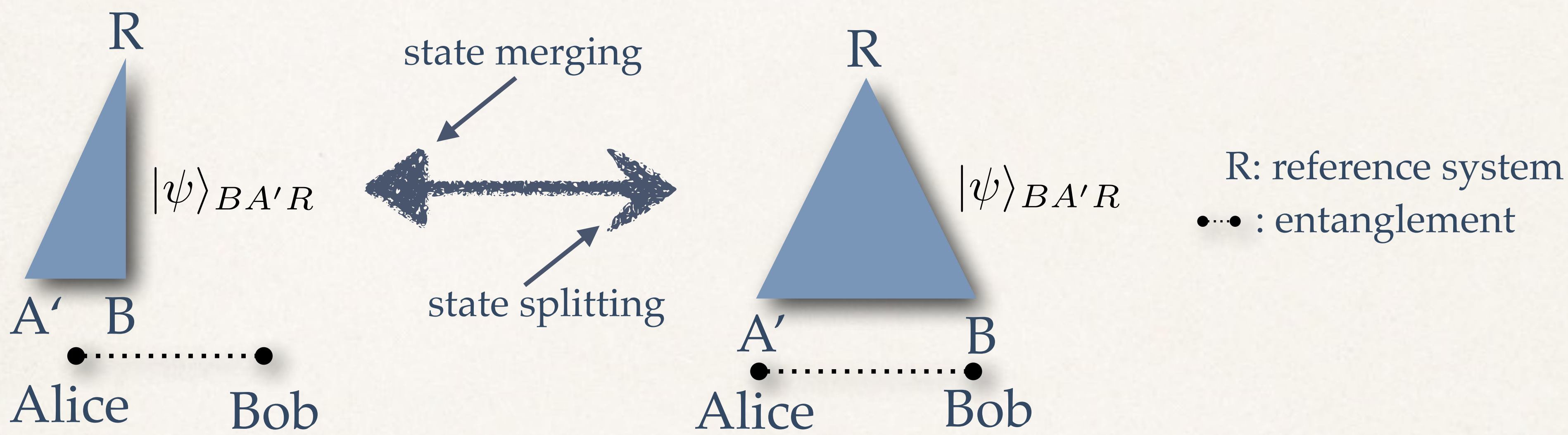
can be accomplished with fidelity better than $(1-\epsilon)$ for $k > m^{1/\epsilon}$ with local isometries at *A* and *B*.
- **Definition:** The *fidelity* between two density matrices ρ and σ is defined as

$$F(\rho, \sigma) = (\mathrm{tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}))^2$$

and it is a notion of distance on the set of density matrices.

[11] Pra, Rc 67:060302(R), 2003

Example: Quantum State Merging/State Splitting



- How much of a given resource is needed to do this?
- Our case: $\sigma_{BA'} \to \sigma_{BA'R} = |\psi\rangle\langle\psi|_{BA'R}$ purification, free entanglement, classical communication to quantify.
- Horodecki et al. [12], $|\psi^{\otimes n}\rangle_{BA'R}$ with classical communication cost c_n :

$$c = \lim_{n\to\infty} \frac{1}{n} c_n = H(\sigma_B) + H(\sigma_R) - H(\sigma_{BR}) = I(B:R)_\sigma$$

- One-shot version, $|\psi\rangle_{BA'R}$ with classical communication cost c_ϵ for an error ϵ [4]:

$$c_\epsilon \cong I^{\epsilon}_{\max}(B:R)_\sigma$$

[12] Nature 436:673-676, 2005

[4] Berta et al., CMP 306(3):579, 2011

Details: The Post-Selection Technique

- Christandl et al. [8]: Let $\mathcal{E}_{A^n}$ and $\mathcal{F}_{A^n}$ be quantum operations that act permutation-covariant on a n-partite system $\mathcal{H}_{A^n} = \mathcal{H}_A^{\otimes n}$. Then

$$\|\mathcal{E}_{A^n} - \mathcal{F}_{A^n}\|_\diamond \leq \text{poly}(n)\|((\mathcal{E}_{A^n} - \mathcal{F}_{A^n}) \otimes \text{id}_{R^n R'})(\zeta_{A^n R^n R'})\|_1$$

where $\zeta_{A^n R^n R'}$ is a purification of the (de Finetti type) state

$$\zeta_{A^n R^n} = \int \omega_{AR}^{\otimes n} d(\omega_{AR})$$

with ω_{AR} a pure state on $\mathcal{H}_A \otimes \mathcal{H}_R$, $\mathcal{H}_R \cong \mathcal{H}_A$, $\mathcal{H}_{R^n} = \mathcal{H}_R^{\otimes n}$ and *d(.)* the measure on the normalized pure states on $\mathcal{H}_A \otimes \mathcal{H}_R$ induced by the Haar measure on the unitary group acting on $\mathcal{H}_A \otimes \mathcal{H}_R$, normalized to $\int d(.) = 1$.

[8] Christandl et al., PRL 102(2):020504, 2009